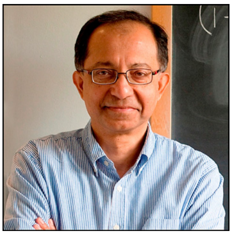
A New and Rather Long Proof of the Pythagorean Theorem by Way of a Proposition on Isosceles Triangles

Kaushik Basu



Kaushik Basu (kb40@cornell.edu) is professor of economics and C. Marks Professor at Cornell University and was, until recently, chief economist and senior vice president of the World Bank. His interests, beyond economics, are in philosophy and art.

There is an abundance of proofs available for Pythagoras's theorem on right-angled triangles, from Pythagoras's own alleged proof in the 6th century B.C., through Euclid's proof [**4**], the proof by Thābit ibn Qurra of Baghdad in the 9th century, and the Indian 12th century mathematician Bhaskara's proof, to the one by the 20th President of the United States, James Garfield [**6**]. (See a related article on p. 361 on a proof attributed to Leonardo da Vinci.)

The aim of this paper is to present a new and rather long proof of this theorem. In doing so, I have to face the inevitable question, "Why?" This is especially so because an important principle of mathematics is brevity. Many times an original, long proof is cast aside with the subsequent discovery of shorter proofs. One example of this within the realm of my own interests is Arrow's celebrated impossibility theorem on voting systems [**1**]. The original proof has been superseded almost purely on grounds of its length by much shorter proofs, e.g., Geanakoplos [**5**] and Sen [**7**].

How then can one justify presenting a new and longer proof of the Pythagorean theorem? To answer this, I invoke another Greek, Constantine Cavafy and his poem *Ithaca*, describing the long journey to Odysseus's home island after the Trojan War. When you reach the island, the poet warns, you are likely to be disappointed, for it will have little new to offer. But he claims that Ithaca's charm is the journey itself [**3**].

> Ithaca gave to you the beautiful journey;
> without her you wouldn't have set upon the road.
> But she has nothing left to give you.

In this article, I take the reader to the familiar, final theorem via two novel propositions or lemmas, one pertaining to isosceles triangles and the other to right-angled triangles. I hope that these lemmas, and especially the one on isosceles triangles, will be of interest in themselves. As Cavafy says in the same poem,

> Hope that the road is a long one.
> Many may the summer mornings be
> when—with what pleasure, with what joy—
> you first put in to harbors new to your eyes.

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To set out the terminology, the Pythagorean theorem says that for any right-angled triangle, in which the hypotenuse has a length of r and the other two sides lengths of a and d, it must be that $r^2 = a^2 + d^2$. We prove the theorem using two lemmas, stated and proved in the next two sections.

The isosceles lemma

Consider an isosceles triangle in which no angle is greater than 90°, such as ABC in Figure 1. Let the two equal sides be of length c and let the length of CB be r. Now from the vertex C drop a perpendicular to the side AB and label the intersection point D; let the length of DB be d.

Next draw a rectangle on the side AB, which has height d (and the other side has length c, of course). Draw an identical rectangle on the side AC. These two rectangles are labeled R_1 and R_2 in Figure 1 right.

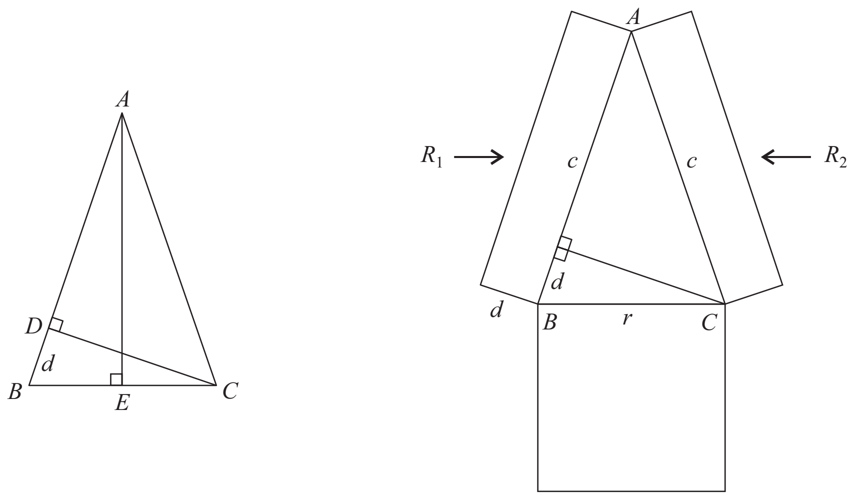

Figure 1.

Lemma (Isosceles). *In the situation described, the sum of the areas of rectangles R_1 and R_2 is equal to the area of the square on the third side, BC. In other words, $r^2 = 2dc$.*

Proof. Continuing with the triangle shown in Figure 1 left, drop a perpendicular from A to the side BC and let E be the point of intersection.

Clearly E bisects BC, hence BE and EC both have length $r/2$. It is easy to see that triangles AEB and CDB are similar. That is, their only (possible) difference is the size. Hence, in terms of lengths, DB is to BC what BE is to AB. That is,

$$\frac{d}{r} = \frac{r/2}{c}$$

from which we conclude $r^2 = 2dc$. ■

Corollary. *In the special case in which the isosceles triangle happens to be right-angled, the isosceles lemma implies the Pythagorean theorem.*

Proof. Consider the angle at vertex A, in Figure 1, to be 90°. For such an isosceles triangle, it is clear that the point D will coincide with A. Hence, $d = c$ in this case. Substituting this into the result of the isosceles lemma gives $r^2 = 2c^2$, matching what the Pythagorean theorem would assert in this case. ■

This is of course not enough for what we are setting out to do in this paper, since the corollary applies to only isosceles right-angled triangles.

Before proceeding further, for the reader wary of the many "equal to's" in geometry, I record Nigel Molesworth's observation from *Down with Skool!* [**8**]:

> To do geom you hav to make a lot of things equal to each other when you can see perfectly well that they don't. This agane is due to Pythagoras and it formed much of his conversation at brekfast.

(The scene ends with the mathematician's wife throwing porridge at him).

The right-angled lemma

Consider a right-angled triangle, such as ABF in Figure 2. Let a be the length of BF and b the length of AB. From the vertex B drop a line to the side AF to that the point of intersection G so that the length of AG is b. In other words, ABG is isosceles. Let c be the length of AF and f the length of FG.

Next draw a rectangle on the side AF and also one on the side AB which each have height f. These are labeled as rectangle R_3 and R_4 in Figure 2.

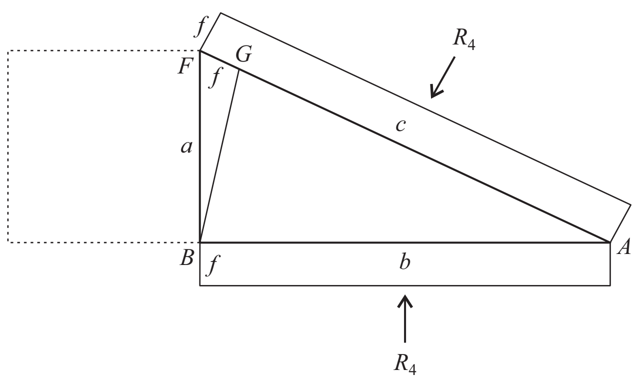

Figure 2.

Lemma (Right-angled). *In the situation described, the sum of the areas of the rectangles R_3 and R_4 is equal to the square on the side FB. That is,* $a^2 = bf + cf$.

Proof. As shown in Figure 3, draw a line parallel to BG through the vertex F and let H be the point of intersection with the line determined by A and B.

Since FH is parallel to GB and ABG is isosceles (by construction), AHF must be isosceles as well. Hence, the length of HB is f. Now draw HG and let J be the point where it intersects FB. Let n be the length of FJ and m the length of JB. Clearly, it follows n is also the length of HJ and m is also the length of JG. Therefore, $n + m = a$.

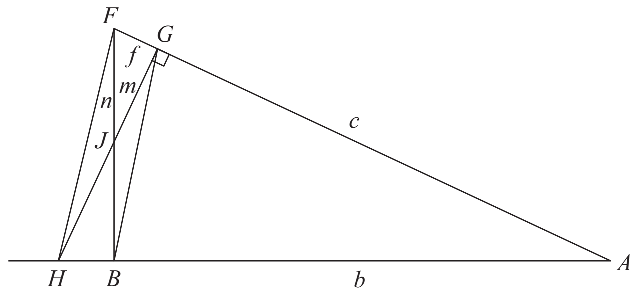

Figure 3.

It is easy to see GFJ and BFA are similar triangles, hence $n/m = c/b$. Combining this with $n + m = a$ gives $n = a - m = a - bn/c$ which is equivalent to

$$n = \frac{ac}{b + c}.$$

Again, since GFJ and BFA are similar, $f/n = a/c$.

From the two previous equations, we know

$$\frac{ac}{b + c} = \frac{fc}{a}$$

which implies $a^2 = bf + cf$. ■

Proof of the Pythagorean theorem

To prove the Pythagorean theorem, consider a right-angled triangle BFH as shown in Figure 4. Let the hypotenuse HF be of length r, let BH be of length d, and BF of length a. We want to prove $r^2 = a^2 + d^2$.

Extend the line HB to the right, to a point A such that the lengths of FA and HA are equal. In other words, such that AHF is isosceles. Let b be the length of AB and c the length of AF.

By the isosceles lemma, we know $r^2 = 2dc$, equivalently $r^2 = dc + d(d + b)$.

By the right-angled lemma, we have $a^2 = db + dc$. These two equations imply $r^2 = a^2 + d^2$. ■

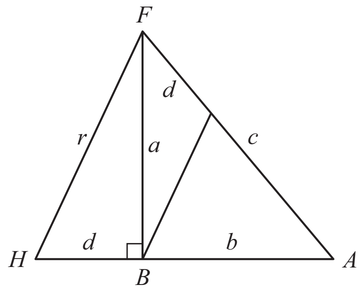

Figure 4.

This establishes the Pythagorean theorem. There are no surprises in the end, not after 2,600 years. But it is hoped that the journey, especially the encounter en route with the isosceles lemma, is of some intrinsic interest to both students and geometers.

Acknowledgment. While the present paper is an exercise in pure geometry, it is the outcome of a long-standing interest of the author in using geometry to understand economics [**2**]. I owe initial interest in this particular topic to a conversation with Arunava Sen. I am grateful to Tito Cordella, Tapan Mitra, and S. Subramanian for reading through the paper and the proofs, and for giving many valuable comments; to an anonymous referee of the journal for useful suggestions; and to Bintao Wang for excellent technical support.

Summary. This paper provides a new, long proof of the Pythagorean theorem. The two lemmas used should be of some intrinsic interest, especially one on isosceles triangles.

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