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Soft sets: An ordinal formulation of vagueness with some applications to the theory of choice

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Abstract: In the theory of fuzzy sets, it is usual to represent the 'degree of belongingness' by a number lying between 0 and 1. This paper develops an alternative framework which dispenses with such cardinal numerical representations of the degree of belongingness, and relies, instead, on an ordinal formulation. The ordinal notion of a vague set is called a soft set. Soft sets are shown to have a tight logical relation with the concept of L-sets introduced earlier by Goguen. The paper also discusses applications of the framework of soft sets to several problems in the theory of choice.

Keywords: Soft set; ordinal notion of the degree of belongingness; vague preferences; theory of choice; consumers' choice; expenditure minimization; preference maximization.

1. Introduction

Our thinking and our everyday language often involve many vague concepts, and the preferences underlying many of our actions are also often vague. Over the last few decades, the theory of fuzzy sets has been widely used for modelling vague concepts in systems analysis, social sciences and artificial intelligence (for a very lucid survey of this literature, see [6]). While the contribution of the theory of fuzzy sets has been fundamental in all these areas, the use of numerical representations of the 'degree of belongingness', and the cardinal (as distinguished from the purely ordinal) aspects of such numerical representations, constitute a somewhat restrictive feature in several contexts. Without denying the obvious importance of the elegant theory that has been developed on the basis of the cardinal numerical representation of the 'degree of belongingness', we feel that there are intuitively persuasive reasons, especially in many areas of the social sciences, for an alternative framework. In this paper, we introduce such a framework; we call it the soft set theoretic structure, and we investigate its formal relations with an earlier ordinal framework due to Goguen [8]. We show that, despite the austere informational basis of the ordinal formulations, they have important applications. We discuss a few applications of our notion of soft sets to the economic theory of consumers' choice and the more abstract theory of preference and choice in general. We believe that not only ordinal formulations of vagueness are logically less demanding than the traditional numerical formulation, and, therefore, conceptually more satisfactory in many areas where numerical representations may seem overly precise, but, also, despite their informational parsimony, the ordinal formulations have sufficient power to be useful in tackling a wide range of problems.

The outline of the paper is as follows. In Section 2 we discuss the intuitive reasons for preferring an ordinal formulation of vagueness in certain fields. In Section 3 we define soft sets and distinguish between alternative types of soft set theoretic structures. In Section 4 we recall the concept of L-sets originally due to Goguen [8]. In Section 5 we show that, despite the different formal structures of our soft sets and Goguen's

L-sets, there exists a tight logical connection between the two structures. In Section 6 we discuss some applications of the notion of soft sets and soft binary relations to the general theory of choice and also to the economic theory of consumers' choice. Section 7 briefly concludes the paper.

2. The case for an ordinal framework

To see some of the limitations of the formulation of vagueness in the standard theory of fuzzy sets, consider a specific vague concept, say, the concept of redness. Clearly, the set of red objects is not well defined. That is, we may not be able to say definitely, for each object, whether it is red or not red. Certain pink or orange objects, for instance, could cause difficulties. Ordinary fuzzy sets seek to capture this vagueness by attaching a number between zero and one to each object; this number represents the *extent* to which the object is red. While this helps to break out of the rigid, dichotomous classification system of conventional set theory, it leads to other conceptual difficulties.

First, a numerical representation of the extent to which different objects are red seems to be rather strong in its informational content. It seems to go against the vagueness of the concept involved here, if the numbers are assumed to have any greater significance than the implied ranking. It is essentially this type of objection which Urquhart [21, p. 108], among others, seems to be voicing when he writes:

> One immediate objection which presents itself to this line of approach is the extremely artificial nature of the attaching of precise numerical values to sentences like '73 is a large number' or 'Picasso's *Guernica* is beautiful'. In fact, it seems plausible to say that the nature of vague predicates precludes attaching precise numerical values just as much as it precludes attaching precise classical truth values.

Note that, if the numerical representation of the different degrees to which different objects are red is no more than just a convenient way of representing the ordering of these different degrees of redness (for a discussion of the problem of representing orderings in terms of real numbers, see [5]), then there would not be anything unique about any given numerical representation. If any given numerical representation does the job of representing the ordering, then so would any positive monotonic transformation of that numerical representation. To use the terminology common in economics, the numerical representation in such cases would be purely ordinal, without any cardinal significance whatsoever. However, it is our understanding that, in the standard theory of fuzzy sets as well as in its applications, numerical degrees of membership are often treated as fully cardinal. Crucial use is made of the cardinal features of the numbers involved, with the consequence that the structure of the theory and the conclusions are not necessarily invariant with respect to an arbitrary positive monotonic transformation of the specific numerical representations used. To cite just one example, the property of connectedness, as defined for a fuzzy binary relation, requires that the number representing the degree to which an ordered pair (x, y) belongs to the fuzzy binary relation and the corresponding number for the ordered pair (y, x) should add up to a number not less than 1. Clearly, this definition uses more information than just the ordering of the degrees to which different ordered pairs belong to the fuzzy binary relation under consideration. Indeed, in the purely ordinal framework it is not clear what it means to require that the sum of two numbers attached to two different ordered pairs should not be less than 1, since, even if it may be true under one numerical representation, it will not be generally true under an arbitrary positive monotonic transformation of that numerical representation.

It is not our intention to deny the case for the cardinal formulation of vagueness in many contexts. The cardinal formulation, though more demanding in its informational requirement, may often yield a richer theory than the more Spartan ordinal framework. We only want to make a weaker claim that in some cases, especially in the theory of preference and choice, the cardinal formulation of vagueness seems to impose an excessively precise structure on the notion of the degree to which an object belongs to the vague set under consideration. Even this weak claim, if accepted, provides considerable motivation for exploring alternative frameworks for discussing vague concepts, which dispense with numerical representations of the 'degree of belongingness', and which rely, instead, on the conceptually more primitive notion of whether an object possesses some vague property to a greater or smaller extent than another object.

The theory of fuzzy sets also raises a second type of intuitive problem as far as it implies a very strong 'completeness' axiom. It entails that, for any two objects, x and y, and for any two vague sets A and B, we can compare the extent to which x belongs to A and the extent to which y belongs to B. Sometimes, this may be, intuitively, too strong an assumption. Consider two vague sets: the set of round objects and the set of red objects. While it may be reasonable to assume comparability of the extents to which different objects are round, and also to assume comparability of the extents to which different objects are red, it may not be equally reasonable to build into the formal framework the comparability of the degree to which an object is round and the degree to which an object is red. The theory of fuzzy sets implies this type of comparison of the 'degrees of belongingness' with respect to different sets, and, from an intuitive point of view, this seems to be excessively rigid. It would seem desirable to have a structure which does not *imply* that all such comparisons across different sets can be made, though, of course, it would be desirable to have a sufficiently flexible formal structure which *permits* such comparisons across sets.

Finally, it is also of considerable formal interest to see how much of our intuition about vague concepts can be captured by a conceptually more economical framework. If, for certain modelling purposes, the weaker ordinal formulation proves to be adequate, then, by the principle of Occam's razor, it would seem undesirable to use, for these purposes at least, the cardinal formulation, even though for other purposes, the richer cardinal formulation may turn out to be more fruitful. The similarity of all this discussion to the cardinal utility versus ordinal utility debate in economics is obvious. Even though the concept of cardinal utility continues to be useful and important in the normative fields of welfare economics and the theory of social choice, the positive economic theory of consumers' choice has opted for the much weaker ordinal utility formulation since the weaker concept proves to be adequate for all the purposes of such positive theory.

3. Soft sets

Throughout this paper, when we use the terms 'set', 'binary relation' or 'function', without using any adjective such as 'fuzzy' 'soft', etc., it is to be understood that we are referring to a crisp set or a crisp binary relation or a crisp function.

Let X be the (crisp) universal set, assumed to be nonempty; and let 2^X be the power set of X.

Definition 3.1. (1) A *soft set theoretic structure* (SST) is an ordered triple $\langle \psi, g, \succcurlyeq \rangle$ such that

$$\psi \text{ is a nonempty set with } \#\psi \geqslant \#2^X; \tag{3.1}$$

$$g \text{ is a one to one function from } 2^X \text{ to } \psi; \tag{3.2}$$

$$\succcurlyeq \text{ is a reflexive and transitive binary relation on } X \times \psi \tag{3.3}$$

(with $\succ$ and $\sim$ denoting respectively the asymmetric and symmetric factors of $\succcurlyeq$).

For all $A \in \psi$, $x, y, z \in X$, and $B \in 2^X$, if $x \in B$ and $y \in X - B$, then

$$(x, g(B)) \succcurlyeq (y, A) \succcurlyeq (y, g(B)) \text{ and } (x, g(B)) \succ (y, g(B)). \tag{3.4}$$

The elements of ψ are called soft sets.

(2) An SST $\langle \psi, g, \succcurlyeq \rangle$ is *proper* iff for all soft sets $A \in \psi$, the restriction of $\succcurlyeq$ to $X \times \{A\}$ is connected (i.e., it is an ordering).

(3) Given an SST $\langle \psi, g, \succcurlyeq \rangle$, two soft sets $A, B \in \psi$ are said to be *identical* (denoted as $A \equiv B$) iff for all $x, y \in X$ for all $D \in \psi$,

$$[(x, A) \succcurlyeq (y, D) \text{ iff } (x, B) \succcurlyeq (y, D)] \tag{3.5a}$$

and

$$[(y, D) \succcurlyeq (x, A) \text{ iff } (y, D) \succcurlyeq (x, B)]. \tag{3.5b}$$

Remark 3.2. (1) Let $\langle \psi, g, \succcurlyeq \rangle$ be a given SST. Then g identifies the exact sets in the set ψ of soft sets. Thus, for all $A \in 2^X$, $g(A)$ can be interpreted to be the exact set A as it figures in the SST ψ. For all $x, y \in X$ and $A, B \in \psi$, '$(x, A) \succcurlyeq (y, B)$' is to be interpreted as 'x

belongs to A to an extent which is at least as great as the extent to which y belongs to B'. Note that $\succcurlyeq$ has not been assumed to be connected over $X \times \psi$; on this point, see Remark 3.2.4.

(2) Intuitively, (3.4) implies that the extent to which an element x of X belongs to a soft set always lies (in a weak sense) between 'belonging definitely' and 'definitely not belonging'.

(3) If the SST is proper then it is always possible to compare the extent to which the different members of X belong to a given soft set, even though it may not be possible to compare the extent to which a member of X belongs to one soft set with the extent to which the same or a different member of X belongs to a different soft set.

(4) Definition 3.1(3) essentially seems to capture the idea that if all members of X belong to two soft sets in 'exactly the same way', then the two soft sets are indistinguishable ('identical'). Given that $\succcurlyeq$ is reflexive, from Definition 3.1(3) it follows that if two soft sets, A and B, are identical, then for all $x \in X$, $(x, A) \sim (x, B)$.

Definition 3.3. Let $\langle \psi, g, \succcurlyeq \rangle$ be an SST.

(1) The SST is of *Type I* iff for all $x, y \in X$ and $A, B \in \psi$,

$$[(x, A) \succcurlyeq (y, B)]$$

implies

$$[(x, A) \sim (x, g(X))$$

$$\text{or } (y, g(B)) \sim (y, g(\phi)) \text{ or } A \equiv B].$$

In this case, all soft sets in ψ are called Type I soft sets.

(2) The SST is of *Type II* iff for all $x, y \in X$ and $A, B \in \psi$, $[(x, A) \succcurlyeq (y, B)$ or $(y, B) \succcurlyeq (x, A)]$. In this case all soft sets in ψ are called Type II soft sets (see Definition 3.1).

Remark 3.4. SSTs of Type I and SSTs of Type II represent two extreme cases. If an SST is of Type I, then for nonidentical soft sets A and B, it is not possible to compare the extent to which x belongs to A with the extent to which y belongs to B unless the comparison is trivial, i.e. unless x definitely belongs to A or x is definitely excluded from A or y definitely belongs to B or y is definitely excluded from B. On the other hand all such comparisons across different soft sets (in addition to comparisons for the same soft set) are possible if the SST is of Type II. The type of SST that one would like to use would, of course, depend on the intuitive assumptions underlying the specific problem under consideration. Suppose we have an object x which is neither 'definitely red' nor 'definitely nonred'. Similarly, suppose we have another object which is neither 'definitely yellow' nor 'definitely nonyellow'. Then, whether one would like to permit comparison of the degree to which x is red with the degree to which y is yellow would depend on the intuitive context, and it is this intuition which would dictate the type of SST one would like to use for the purpose of modelling. It is, of course, obvious that one need not use either of the polar types introduced in Definition 3.3; instead, one may choose a hybrid type where some, but not all, nontrivial comparisons across different soft sets are permitted.

Definition 3.5. Let $\langle \psi, g, \succcurlyeq \rangle$ be a proper SST. Then $\langle \psi, g, \succcurlyeq \rangle$ is said to be *complete* iff for every $A \in [\psi - \{g(X), g(\phi)\}]$, for every $x \in X$ and for every ordering T over

$$(X \times \{A\}) \cup \{(x, g(X)), (x, g(\phi))\},$$

such that $(x, g(X))$ is a T-greatest element of

$$(X \times \{A\}) \cup \{(x, g(X)), (x, g(\phi))\}$$

and $(x, g(\phi))$ is a T-least element of

$$(X \times \{A\}) \cup \{(x, g(X)), (x, g(\phi))\},$$

there exists $B \in \psi$ such that for all $y, z \in X$,

$$[(y, A)\ T\ (z, A) \text{ iff } (y, B) \succcurlyeq (z, B)];$$

$$[(x, g(X))\ T\ (y, A) \text{ iff } (x, g(X)) \succcurlyeq (y, B)];$$

$$[(y, A)\ T\ (x, g(X)) \text{ iff } (y, B) \succcurlyeq (x, g(X))];$$

$$[(x, g(\phi))\ T\ (y, A) \text{ iff } (x, g(\phi)) \succcurlyeq (y, B)];$$

and

$$[(y, A)\ T\ (x, g(\phi)) \text{ iff } (y, B) \succcurlyeq (x, g(\phi))]$$

(see Definition 3.1).

Remark 3.6. If a proper SST $\langle \psi, g, \succcurlyeq \rangle$ of Type I is complete, then it would not be possible to 'expand' the SST by 'additing' to ψ a soft set which is not identical to one of the soft sets already in ψ.

Notation 3.7. Let $\langle \psi, g, \succcurlyeq \rangle$ be a given SST. Then $\rho(\psi)$ denotes the cardinality of an exact subset ψ^* of ψ such that for all $A \in \psi$, there exists $B \in \psi^*$ such that $A \equiv B$, and for all $A, B \in \psi^*$, not $(A \equiv B)$.

Notation 3.8. Let $\pi(X)$ denote the set of all ordered nontrivial partitions of X. (Nontriviality excludes the partition $\{X, \phi\}$ from $\pi(X)$.)

The proof of the following proposition is straightforward and is therefore omitted.

Proposition 3.9. *Suppose X is a finite set, and suppose $\langle \psi, g, \succcurlyeq \rangle$ is a proper SST of Type I. Then $\rho(\psi) \leq 4(\#\pi(X)) + 3$. Further, if $\langle \psi, g, \succcurlyeq \rangle$ is also complete, then $\rho(\psi) = 4(\#\pi(X)) + 3$ (see Definitions* 3.1 *and* 3.2, *and Notations* 3.7 *and* 3.8).

4. L-Sets

In this section, we recall the notion of L-sets which is due to Goguen [8].

Definition 4.1. (1) An L-set theoretic structure on X (LST) is an ordered triple $\langle \xi, \succcurlyeq_*, L \rangle$ where L is a nonempty set with $\#L \geq 2$; $\succcurlyeq_*$ is a reflexive, transitive and antisymmetric binary relation over L; ξ is a nonempty set of functions from X to L; there exist a unique $\succcurlyeq_*$-greatest element $\bar{d}$ in L and a unique $\succcurlyeq_*$-least element $\underline{d}$ in L; and for all $A \in 2^X$, there exists $f \in \xi$ such that for all $x \in A$, $f(x) = \bar{d}$, and for all $y \in X - A$, $f(y) = \underline{d}$. The element of ξ are called *L-sets*.

(2) An LST $\langle \xi, \succcurlyeq_*, L \rangle$ is *proper* iff for every $f \in \xi$, the restriction of $\succcurlyeq_*$ to $f(X)$ is an ordering.

(3) An LST $\langle \xi, \succcurlyeq_*, L \rangle$ is *complete* iff ξ is the set of all possible functions from X to L.

(4) An LST $\langle \xi, \succcurlyeq_*, L \rangle$ is *normal* iff L is a lattice under the meet and joint operations induced by $\succcurlyeq_*$ on L.

Remark 4.2. (1) Intuitively, L is the set of all possible 'degrees of belongingness'. $\bar{d}$ corresponds to the notion of 'definitely belonging' and $\underline{d}$ corresponds to the notion of 'definitely not belonging'.

(2) Intuitively again, the LST is proper iff it is possible to compare the extents to which any two elements of X belong to any given L-set in the LST.

5. The relation between soft set theoretic structures and L-set theoretic structures

In this section we explore the close logical relation that exists between SSTs on the one hand and LSTs on the other.

Notation 5.1. $\mathscr{A}$ denotes the set of all SSTs and $\mathscr{B}$ denotes the set of all LSTs.

Notation 5.2. (1) Let T_α denote a function from $\mathscr{A}$ to $\mathscr{B}$, defined as follows. For every $\langle \psi, g, \succcurlyeq \rangle \in \mathscr{A}$, let $T_\alpha(\langle \psi, g, \succcurlyeq \rangle)$ be $\langle E, (\succcurlyeq_*), Q \rangle$ where

Q is the set of equivalence classes induced by $\sim$, the symmetric component of $\succcurlyeq$; (5.1)

$(\succcurlyeq_*)$ is a reflexive, transitive and antisymmetric binary relation on Q such that for all $q, q' \in Q$,

$$q(\succcurlyeq_*)q' \text{ iff for all } (x, A) \in q \text{ and all } (x', A') \in q', (x, A) \succcurlyeq (x', A'); \tag{5.2}$$

E is the set of all functions $e: X \to Q$ such that for some $A \in \psi$,

$$\text{for all } x \in X,\ e(x) = [(x, A)], \tag{5.3}$$

where $[(x, A)]$ is the equivalence class of (x, A) in Q.

(2) Let T_β be a function from $\mathscr{B}$ to $\mathscr{A}$ defined as follows. For every $\langle \xi, \succcurlyeq_*, L \rangle \in \mathscr{B}$, let $T_\beta(\langle \xi, \succcurlyeq_*, L \rangle)$ be $\langle \xi, h, (\succcurlyeq) \rangle$, where h is a one-to-one function from 2^X to ξ such that for all $A \in 2^X$, for all $x \in A$, and for all $y \in (X - A)$,

$h(A)(x)$ is the unique $\succcurlyeq_*$-greatest element in L and $h(A)(y)$ is the unique $\succcurlyeq_*$-least element in L; (5.4)

and $(\succcurlyeq)$ is a reflexive and transitive binary relation over $X \times \xi$ such that for all $x, x' \in X$ and all $f, f' \in \xi$,

$$(x, f)(\succcurlyeq)(x', f') \text{ iff } f(x) \succcurlyeq_* f'(x'). \tag{5.5}$$

(See Notation 5.1.)

Definition 5.3. (1) For all $\langle \psi, g, \geqslant \rangle$, $\langle \psi', g', \geqslant' \rangle \in \mathscr{A}$, an *S-morphism* is a one-to-one and onto function S from ψ to ψ' such that for all $x, y \in X$ and all $A, B \in \psi$. $(x, A) \geqslant (y, B)$ iff $(x, S(A)) \geqslant' (y, S(B))$. If such an S-morphism exists, then $\langle \psi, g, \geqslant \rangle$ and $\langle \psi', g', \geqslant' \rangle$ are said to be *S-morphic.*

(2) For all $\langle \xi, \geqslant_*, L \rangle$, $\langle \xi', \geqslant'_*, L' \rangle \in \mathscr{B}$, an *L-morphism* is a one-to-one and onto function l from ξ to ξ' such that for all $x, y \in X$, and all $f, f' \in \xi$,

$$f(x) \geqslant_* f'(y) \text{ iff } (l(f))(x) \geqslant'_* (l(f'))(y).$$

If such an L-morphism exists, then $\langle \xi, \geqslant_*, L \rangle$ and $\langle \xi', \geqslant'_*, L' \rangle$ are said to be *L-morphic* (see Notation 5.1).

Remark 5.4. S-morphic (resp. L-morphic) SSTs (resp. LSTs) are essentially the same except for possible renaming of the soft sets (L-sets).

Remark 5.5. 'Being S-morphic to' (resp. 'being L-morphic to') is an equivalence relation on $\mathscr{A}$ (resp. $\mathscr{B}$).

Notation 5.6. Let the set of equivalence classes generated by the equivalence relation 'being S-morphic to' (resp. 'being L-morphic to') defined over $\mathscr{A}$ (resp. $\mathscr{B}$) be denoted by Eq($\mathscr{A}$) (resp. Eq($\mathscr{B}$)). Let α be a function from $\mathscr{A}$ to Eq($\mathscr{A}$) such that for all $\theta \in \mathscr{A}$, $\alpha(\theta) = [\theta]$ where $[\theta]$ denotes the equivalence class of θ, defined by the equivalence relation 'being S-morphic to' over $\mathscr{A}$. Similarly, let β be a function from $\mathscr{B}$ to Eq($\mathscr{B}$) such that for all $\delta \in \mathscr{B}$, $\beta(\delta) = [\delta]$ where $[\delta]$ denotes the equivalence class of δ, defined by the equivalence relation 'being L-morphic to' over $\mathscr{B}$ (see Notation 5.1 and Definition 5.3).

The following proposition is obvious.

Proposition 5.7. *Let θ, $\theta' \in \mathscr{A}$ and δ, $\delta' \in \mathscr{B}$. Then θ and θ' are S-morphic iff $T_\alpha(\theta)$ and $T_\alpha(\theta')$ are L-morphic. Also, δ and δ' are L-morphic iff $T_\beta(\delta)$ and $T_\beta(\delta')$ are S-morphic (see Notations* 5.1 *and* 5.2 *and Definition* 5.3).

Notation 5.8. In view of Proposition 5.7, we can introduce a mapping $\underline{T}_\alpha$ from Eq($\mathscr{A}$) to Eq($\mathscr{B}$), and also a mapping $\underline{T}_\beta$ from Eq($\mathscr{B}$) to Eq($\mathscr{A}$) as follows: (i) for all $a \in \text{Eq}(\mathscr{A})$, $\underline{T}_\alpha(a) = \beta(T_\alpha(\theta))$ for some $\theta \in a$; and (ii) for all $b \in \text{Eq}(\mathscr{B})$, $\underline{T}_\beta(b) = \alpha(T_\beta(\delta))$ for some $\delta \in b$ (see Notations 5.1, 5.2 and 5.6).

The logical relation that exists between SSTs

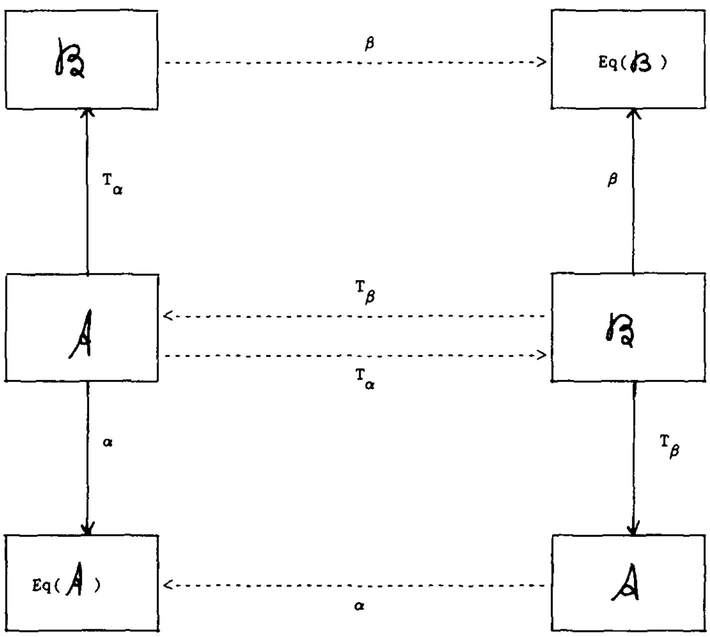

Fig. 1. Relation between SSTs and LSTs.

on the one hand and LSTs on the other can now be summarized in terms of the diagram shown in Figure 1.

6. Some applications of soft set theoretic structures to the theory of choice

In view of the close logical relation between SSTs and LSTs, demonstrated in the preceding section, it is clear that whether we choose to work in the framework of SSTs or in the framework of LSTs is mainly a matter of convenience in the context of the specific problem under consideration. In this section we discuss some applications of SSTs to several problems in the theory of choice on the basis of vague preferences.

Some of the intuitive issues explored in this section have been discussed earlier in the cardinal framework of standard fuzzy set theory; see, for example, Basu [4], Dutta et al. [7] and Panda and Patanaik [13]. For discussion, again in the cardinal fuzzy set theoretic framework, of other related problems, see Orlovsky [11], Ovchinnikov [12], Switalski [20], Barrett and Pattanaik [2] and Barrett et al. [3] among others.

Definition 6.1. Let $\langle \Omega, q, \succcurlyeq \rangle$ be a proper and complete SST on X^2. Each soft set in Ω will be called a *soft binary relation* (SBR) over X. An *exact soft binary relation* (ESBR) over X is an SBR R such that $R = q(Y)$ for some exact subset Y of X^2 (see Definitions 3.1, 3.3 and 3.5).

Remark 6.2. It is obvious that if R is an ESBR, then, for all $x, y \in X$, either $R(x, y) \sim (q(X^2)(x, x)$ or $R(x, y) \sim (q(\phi))(x, x)$. It is also clear that an ESBR is essentially an exact binary relation redefined in the framework of soft sets.

Notation 6.3. To economise notation, for all $R \in \Omega$, and all $(x, y) \in X^2$, we write $R(x, y)$ rather than $((x, y), R)$. Also we denote $q(X^2)$ by $\bar{R}$ and $q(\phi)$ by $\underline{R}$.

Notation 6.4. Let $J \subseteq X^2$. If there exists a greatest lower bound (defined in terms of $\succcurlyeq$) of J, then we let it be denoted by $m(J)$.

Definition 6.5. A *soft ordering* (SO) over X, is an SBR R over X, which satisfies:

(a) reflexivity: for all $x \in X$,

$$R(x, x) \sim \bar{R}(x, x); \tag{6.1}$$

(b) connectedness: for all distinct $x, y \in X$, if $R(y, x) \sim \underline{R}(x, x)$, then

$$R(x, y) \sim \bar{R}(x, x); \tag{6.2}$$

(c) max–min transitivity: for all $x, y, z \in X$,

$$R(x, z) \succcurlyeq m(\{R(x, y), R(y, z)\}). \tag{6.3}$$

An *exact soft ordering* (ESO) over X is an ESBR which satisfies (6.1), (6.2) and (6.3) (see Definition 6.1 and Notations 6.3 and 6.4).

In Proposition 6.6, we note a property of soft orderings which will be useful later.

Proposition 6.6. *Let R be a soft ordering over X. For all, $x, y, z \in X$, if*

$$[R(x, y) \sim \bar{R}(x, x) \,\&\, R(y, x) \sim \underline{R}(x, x) \,\&\, (y, z) \succ \underline{R}(x, x)],$$

then

$$[R(x, z) \sim \bar{R}(x, x) \,\&\, R(z, x) \sim \underline{R}(x, x)];$$

and if

$$[R(x, y) \succ \underline{R}(x, x) \,\&\, R(y, z) \sim \bar{R}(x, x) \,\&\, R(z, y) \sim \underline{R}(x, x)],$$

then

$$[R(x, z) \sim \bar{R}(x, x) \,\&\, R(z, x) \sim \underline{R}(x, x)].$$

Proof. Let

$$[R(x, y) \sim \bar{R}(x, x) \,\&\, R(y, x) \sim \underline{R}(x, x) \,\&\, R(y, z) \succ \underline{R}(x, x)].$$

Suppose $R(z, x) \succ \underline{R}(x, x)$. Then, from $R(y, z) \succ \underline{R}(x, x)$, by max–min transitivity, it follows that $R(y, x) \succ \underline{R}(x, x)$. This contradicts $[R(y, x) \sim \underline{R}(x, x)]$. Hence, we must have $R(z, x) \sim \underline{R}(x, x)$. Therefore, from the connectedness of R, $R(x, z) \sim \bar{R}(x, x)$. The second part of the proposition can be proved similarly. □

Whenever we shall need to consider an agent's vague preferences over X, those vague preferences will be formally represented by a soft ordering R over X, defined with reference to a given proper and complete SST $\langle \Omega, q, \succcurlyeq \rangle$ on X^2. The interpretation of $[R(x, y) \succcurlyeq R(z, w)]$ will be that the extent to which (x, y) belongs to R (i.e., the extent to which x is at least as good as y) is at least as great as the extent to which (z, w) belongs to R (i.e., the extent to which z is at least as good as w).

Since we shall discuss the choices made by the agent, we introduce the notion of the agents's choice function which seeks to formalize the choice behaviour of the agent.

Definition 6.7. A *choice function* is a function $C: K \to 2^X$, such that $\emptyset \neq K \subseteq (2^X - \{\emptyset\})$ and, for all $A \in K$, $\emptyset \neq C(A) \subseteq A$. A choice function is said to be *unrestricted* iff its domain is the set of all nonempty subsets of X.

Remark 6.8. Intuitively, $C(A)$ denotes the set of alternatives 'chosen' when A is the set of available alternatives. K represents the set of 'choice problems' of the agent, which are under consideration.

In the rest of this section, C denotes the agents's choice function. The domain of the choice function will be denoted by K.

One problem, which has received considerable attention in the theory of exact preferences and choice (see, for example, Arrow [1], Herzberger [9], Houthakker [10], Richter [14, 16] and Sen [17]), is the problem of whether the agent's choices can be explained or 'rationalized' in terms of a crisp ordering (note that, in our framework, a crisp ordering is accommodated as an exact soft ordering). One can pose a similar problem in the context of soft preferences: given the agent's choice function C, one can ask whether there exists a soft ordering which can induce C in some plausible sense. However, here we face a conceptual problem. There is an intuitively obvious sense in which one can think of the choice function as being rationalizable in terms of an exact soft ordering: given an exact soft ordering R and the choice function C, C is said to be rationalizable in terms of R (or, induced by R) iff for all $A \in K$, $C(A) = \{x \in A \mid R(x, y) = \bar{R}(x, x)$ for all $y \in A\}$. Unfortunately, in the case of nonexact soft orderings, there is no such straightforward notion of rationalizability of the choice function. If the soft ordering R is nonexact, then there are several different notions of rationalizability of the choice function in terms of R, and none of these seems to be obviously more plausible than the others. Thus, when the soft orderings are not constrained to be exact, one can think of several alterative assumptions about the choice function C, each of which corresponds to some notion of the rationalizability of C in terms of a soft ordering. We consider three such asssumptions below.

Note that in judging the relative intuitive appeal of these alternative notions of rationalizability in terms of soft preferences, it would be important to have axiomatizations, similar to the axiomatizations in the 'rationalizability literature' on crisp preferences and choice (see, for example, Arrow [1], Herzberger [9], Houthakker [10], Richter [14, 15, 16], Sen [17] and Suzumura [19]). This, however, needs detailed independent investigation.

Notation 6.9. Let R be a given soft ordering over X. Then, for all $x^*, y^* \in X$, let

$$H^1(A, R, x^*, y^*) = \{x \in X \mid \text{for all } y \in A,\ R(x, y) \succcurlyeq R(x^*, y^*)\};$$

and let

$$H^2(A, R) = \{x \in A \mid \text{for all } y \in A,\ R(x, y) \succcurlyeq R(y, x)\}.$$

Lastly let

$$H^3(A, R) = \{x \in A \mid m(\{x\} \times A) \succcurlyeq m(\{y\} \times A) \text{ for all } y \in A, \text{ and } m(\{x\} \times A) \succ \underline{R}(x, x)\},$$

if, for all $x \in X$, there exists a greatest lower bound of $\{x\} \times A$, defined in terms of $\succcurlyeq$. If for some $x \in X$, there does not exist a greatest lower bound of $\{x\} \times A$, defined in terms of $\succcurlyeq$, then let $H^3(A, R)$ denote $\emptyset$.

Assumption 6.10. There exists a soft ordering such that for some $(x^*, y^*) \in X^2$, $R(x^*, y^*) \succ \underline{R}(x^*, y^*)$, and for all $A \in K$,

$$C(A) = H^1(A, R, x^*, y^*). \quad (6.4)$$

(See Notation 6.9.)

Assumption 6.11. These exists a soft ordering R such that for all $A \in K$,

$$C(A) = H^2(A, R). \tag{6.5}$$

(See Notation 6.9.)

Assumption 6.12. There exists a soft ordering R such that for all $A \in K$,

$$C(A) = H^3(A, R). \tag{6.6}$$

(See Notation 6.9.)

Remark 6.13. Given a soft ordering R, (6.4) represents a principle of 'satisficing' (see [18] for a discussion of 'satisficing'). If (6.4) holds, then, intuitively, the agent fixes a pair of alternatives (x^*, y^*) such that $R(x^*, y^*) > \underline{R}(x^*, x^*)$. Then, given a feasible set of alternatives, A, the agent chooses all those alternatives x in A, such that, for all $y \in A$, the extent to which x is at least as good as y is at least as great as the extent to which x^* is at least as good as y^*. The extent to which x^* is at least as good as y^* sets the level with reference to which 'satisficing' takes place. Note that a stringent application of this type of choice rule arises when $R(x^*, y^*) \sim \bar{R}(x^*, y^*)$, i.e. when x^* is definitely at least as good as y^*. Equation (6.5) refers to a simple form of binary comparisons. Given $A \in K$, $x \in A$ is chosen if and only if, for all $y \in A$, the extent to which x is at least as good as y is at least as great as the extent to which y is at least as good as x. Equation (6.6) embodies a type of max–min principle.

Note that since, by the definition of the choice function C, $C(A) \neq \emptyset$ for all $A \in K$, and since, for all $A \in K$ and all $x \in A$, if there does not exist a greatest lower bound (defined in terms of $\succcurlyeq$) of the set $\{x\} \times A$, then $H^3(A, R) = \emptyset$, it is clear that (6.6) can hold only if, for all $A \in K$ and all $x \in A$, there exists a greatest lower bound of $\{x\} \times A$ (in terms of $\succcurlyeq$). If the set is finite, then this condition is clearly fulfilled. However, when the feasible set of alternatives can have an infinite number of elements (as in the theory of consumers' choice discussed below), additional assumptions may be required to ensure that this condition is fulfilled.

Proposition 6.14. *Suppose the choice function of the agent is unrestricted. Then the agent satisfies Assumption* 6.10 *iff there exists an exact soft ordering R' such that for all $A \in K$, $(C(A) = \{x \in A \mid R'(x, y) \sim \bar{R}(x, x)$ for all $y \in A\}$ (see Definition* 6.7).

Proof. The sufficiency part of the proof is obvious. To prove the necessity part, suppose there exists $x^*, y^* \in X$ and a soft ordering R such that $R(x^*, y^*) > \underline{R}(x^*, y^*)$ and for all $A \in K$, (6.4) holds. Then define an exact soft binary relation R' as follows: for all $x, y \in K$,

$$R'(x, y) \sim \bar{R}(x, x) \quad \text{iff} \quad R(x, y) \succcurlyeq R(x^*, y^*)$$

and

$$R'(x, y) \sim \underline{R}(x, y) \quad \text{iff} \quad R(x^*, y^*) > R(x, y).$$

Given that R is reflexive, for all $x \in X$, $R(x, x) \succcurlyeq R(x^*, y^*)$, and hence $R'(x, x) \sim \bar{R}(x, x)$. Thus, R' is reflexive. For all $x, y \in X$, since $\{x, y\} \in K$, either $R(x, y) \succcurlyeq R(x^*, y^*)$ or $R(y, x) \succcurlyeq R(x^*, y^*)$. Hence either $R'(x, y) \sim R(x, x)$ or $R'(y, x) \sim R(x, x)$. Thus, R' is connected. The max–min transitivity of R' follows directly from the definition of R' and the max–min transitivity of R. Thus R' is an exact soft ordering. All that remains to be shown is that for all $A \in K$,

$$\{x \in A \mid R'(x, y) \sim \bar{R}(x, x) \text{ for all } y \in A\} = H^1(A, R, x^*, y^*).$$

However, this is obvious from the definition of R'. □

Proposition 6.15. *The agent satisfies Assumption 6.11 iff there exits an ESBR R' such that R' satisfies reflexivity* (6.1), *connectedness* (6.2) *and exact quasitransitivity*:

for all $x, y, z \in X$, if

$$[R'(x, y) \sim \bar{R}(x, x) \,\&\, R'(y, x) \sim \underline{R}(x, x) \,\&\, R'(y, z) \sim \bar{R}(x, x) \,\&\, R'(z, y) \sim \underline{R}(x, x)],$$

then

$$[R'(x, z) \sim \bar{R}(x, x) \,\&\, R'(z, x) \sim \underline{R}(x, x)], \tag{6.7}$$

and for all $A \in K$,

$$C(A) = \{x \in A \mid \text{for all } y \in A,\ R'(x, y) \sim \bar{R}(x, x)\}.$$

Proof. (I) Necessity: Suppose the agent satisfies

Assumption 6.11. Define an ESBR R' such that for all $x, y \in X$,

$$[R'(x, y) \sim \bar{R}(x, x) \text{ iff } R(x, y) \succcurlyeq R(y, x)]$$

and

$$[R'(x, y) \sim \underline{R}(x, x) \text{ iff } R(y, x) \succ R(x, y)].$$

Since the SST $\langle \Omega, q, \succcurlyeq \rangle$ over X^2 is proper, for all $x, y \in X$, either $R(x, y) \succcurlyeq R(y, x)$ or $R(y, x) \succcurlyeq R(x, y)$. Hence, for all $x, y \in X$, either $R'(x, y) \sim \bar{R}(x, x)$ or $R'(y, x) \sim \bar{R}(x, x)$. Therefore, R' is reflexive and connected. To show that R' satisfies (6.7), suppose, for some $x, y, z \in X$,

$$[R'(x, y) \sim \bar{R}(x, x) \,\&\, R'(y, x) \sim \underline{R}(x, x) \,\&\, R'(y, z) \sim \bar{R}(x, x) \,\&\, R'(z, y) \sim \underline{R}(x, x)].$$

Then, by the definition of R', $R(x, y) \succ R(y, x) \,\&\, R(y, z) \succ R(z, y)$. We show that $R(x, z) \succ R(z, x)$. Given that $\langle \Omega, q, \succcurlyeq \rangle$ is proper, either $R(x, y) \succcurlyeq R(y, z)$ or $R(y, z) \succcurlyeq R(x, y)$. Suppose $R(x, y) \succcurlyeq R(y, z)$. Then, given the max–min transitivity of R,

$$\begin{aligned} R(x, z) &\succcurlyeq m(\{R(x, y), R(y, z)\}) \\ &= R(y, z) \succ R(z, y) \\ &\succcurlyeq m(\{R(z, x), R(x, y)\}). \end{aligned} \tag{6.8}$$

If

$$m(\{R(z, x), R(x, y)\}) = R(x, y),$$

then, from (6.8), we would have $R(y, z) \succ R(x, y)$, which would contradict our assumption that $R(x, y) \succcurlyeq R(y, z)$. Hence

$$m(\{R(z, x), R(x, y)\}) = R(z, x),$$

and hence, by (6.8), we have $R(x, z) \succ R(z, x)$. Similarly, it can be shown that when $R(y, z) \succcurlyeq R(x, y)$, we would again have $R(x, z) \succ R(z, x)$. Thus in all cases, we have $R(x, z) \succ R(z, x)$. Then it follows that $R'(x, z) \sim \bar{R}(x, x)$ and $R'(z, x) \sim \underline{R}(x, x)$. Thus, R' satisfies (6.7). Consider any $A \in K$. By Assumption 6.11, $C(A) = H^2(A, R)$, and, by the definition of R',

$$H^2(A, R) = \{x \in A \mid R'(x, y) \sim \bar{R}(x, x) \text{ for all } y \in A\}.$$

Hence

$$C(A) = \{x \in A \mid R'(x, y) \sim \bar{R}(x, x) \text{ for all } y \in A\}.$$

This completes the proof of necessity.

(II) Sufficiency: Suppose there exists an ESBR R' which satisfies reflexivity, connectedness and (6.7), and suppose that for all $A \in K$,

$$C(A) = \{x \in A \mid \text{for all } y \in A, \; R'(x, y) \sim \bar{R}(x, x)\}.$$

If for all $z, w \in X$, $R'(w, z) \sim \bar{R}(z, z)$, then it is clear that R' satisfies max–min transitivity and is a soft ordering, and that $C(A) = H^2(A, R')$ for all $A \in K$. Suppose there exist $z, w \in X$, such that not $[R'(w, z) \sim \bar{R}(z, z)]$. Consider such z and w. Since R' is an ESBR satisfying (6.2), it is clear that $R'(z, w) \sim \bar{R}(z, z)$ and $R'(w, z) \sim \underline{R}(z, z)$. Construct an SBR R such that for all $x \in X$, $R(x, x) \sim \bar{R}(x, x)$; $\bar{R}(z, z) \succ R(z, w) \succ R(w, z) \succ \underline{R}(z, z)$; and for all distinct $x, y \in X$, (if $R'(x, y) \sim (\bar{R}(x, x)$ and $R'(y, x) \sim \underline{R}(x, x)$, then $R(x, y) \sim R(z, w)$ and $R(y, x) \sim R(w, z))$ and (if $R'(x, y) \sim R'(y, x) \sim \bar{R}(x, x)$, then $R(x, y) \sim R(y, x) \sim R(z, w)$). (Note that R' being connected, we cannot have $R'(x, y) \sim R'(y, x) \sim \underline{R}(x, x)$.) Given the construction of R, and given that R' satisfies reflexivity, connectedness and (6.7), it can be easily shown that R is a soft ordering. It is also clear that for all $x, y \in X$, $R'(x, y) \sim \bar{R}(x, x)$ iff $R(x, y) \succcurlyeq R(y, x)$ and hence

$$C(A) = \{x \in A \mid R(x, y) \succcurlyeq R(y, x)\}$$

for all $A \in K$. □

Remark 6.16. Proposition 6.14 shows that if the agent has an unrestricted choice function and if he satisfies Assumption 6.10, then his choice behaviour can be explained in the simpler framework of exact orderings. Similarly, Proposition 6.15 shows that if the agent satisfies Assumption 6.11, then his choice behaviour can be explained in terms of an exact binary weak preference relation, though the exact binary weak preference relation may not necessarily be an ordering (note that in Proposition 6.15 we do not use the assumption of an unrestricted choice function). This, however, is not true in the case of Assumption 6.12 as the following example shows.

Example 6.17. Let $X = \{x, y, z\}$ and let R be an SO such that

$$R(x, x) \sim R(y, y) \sim R(z, z) \sim \bar{R}(x, x) \\ > R(x, y) > R(y, x) \sim R(x, z) \sim R(y, z) \\ > R(z, y) \sim R(z, x) > \underline{R}(x, x).$$

Let C be an unrestricted choice function such that $C(\{x, y\}) = \{x\}$; $C(\{y, z\}) = \{y\}$; $C(\{x, z\}) = \{x\}$; and $C(X) = \{x, y\}$). Then the agent satisfies Assumption 6.12 but it can be easily checked that there does not exist any ESBR R' such that for all $A \in K$, $C(A) = \{\bar{x} \in A \mid R'(\bar{x}, \bar{y}) \sim \bar{R}(\bar{x}, \bar{x})$ for all $\bar{y} \in A\}$.

We now consider a specific interpretation of the agent's choice problem. We assume that the agent is a competitive consumer.

Notation 6.18. Let n be the number commodities. Let $\mathbb{R}_+$ and $\mathbb{R}_{++}$ be, respectively, the set of all non-negative real numbers and the set of all positive real numbers. The price vectors will be denoted by $p, p', \ldots \in \mathbb{R}^n_{++}$; and the levels of the consumer's wealth will be denoted by $W, W', \ldots \in \mathbb{R}_{++}$. The consumption set of the consumer is given by $X = \mathbb{R}^n_+$. Given prices p and wealth W, the budget set of the consumer is denoted by

$$A(p, W) = \{x \in X \mid p \cdot x \leqslant W\}.$$

Let the consumer's choice function be $C: K \to (2^X - \{\emptyset\})$, where

$$\emptyset \neq K \subseteq \{A \subseteq X \mid A = A(p, W) \text{ for some price vector } p \text{ and some wealth } W\}.$$

The consumer has a soft ordering R over X, defined with reference to a proper and complete SST $\langle \Omega, q, \succcurlyeq \rangle$ on X^2.

How much of the standard theory of consumers' behaviour survives when we switch to the broader framework of soft orderings? This is a question of considerable interest. We investigate some aspects of this problem here. We concentrate on two fundamental, though elementary, results in the standard theory of consumers' choice. The first is the duality between preference maximization and expenditure minimization in the framework of soft orderings (see [5, pp. 67–71] for a discussion of this duality in the context of crisp preference relations). Secondly, in the same framework, we prove the basic 'substitution theorem' which shows that, if the price of some commodity falls (other prices remaining the same) and the consumer's nominal wealth is simultaneously adjusted so that in the new price–wealth situation he can just buy the commodity bundle he was buying in the original price–wealth situation, then he will not buy less of the commodity the price of which has fallen (see [22, Chap. 3]).

Assumption 6.19 (*Strict monotonicity*). For all $x, y \in X$, if $x > y$, then $R(x, y) \sim \bar{R}(x, x)$ and $R(y, x) \sim \underline{R}(x, x)$.

Remark 6.20. Strict monotonicity implicitly assumes that all the commodities are desirable. If we assume that all the commodities are desirable, then, while the consumer may have a vague ranking of two commodity bundles one of which contains more of some commodity and less of some other commodity than the other, the consumers' ranking over two commodity bundles one of which is unambiguously bigger than the other is unlikely to be vague.

Assumption 6.21 (*Local nonsaturation*). For every $x \in X$, and every neighbourhood τ of x, there exists $y \in \tau$ such that $R(y, z) \sim \bar{R}(x, x)$ and $R(x, y) \sim \underline{R}(x, x)$.

Remark 6.22. Strict monotonicity implies local nonsaturation though the converse is not necessarily true.

Assumption 6.23. For all $A \in K$ and all $x, y \in A$, if $R(x, y) \sim \bar{R}(x, x)$ and $R(y, x) \sim \underline{R}(x, x)$, then $y \notin C(A)$.

Remark 6.24. Assumption 6.23 is a very weak assumption which is implied by each of (6.4), (6.5) and (6.6). Essentially, it says that the consumer does not choose a consumption bundle which is strictly less preferred, in the exact sense, than another available consumption bundle.

Proposition 6.25. *Suppose the consumer satisfies Assumptions* 6.21 *and* 6.23. *Suppose for some price vector p and wealth W, and for some* $x \in X$, $A(p, W) \in K$ *and* $x \in C(A(p, W))$. *Then* $p \cdot x \leqslant W \leqslant p \cdot y$ *for all* $y \in X$ *such that* $R(y, x) > \underline{R}(x, x)$.

Proof. Suppose the hypothesis of the proposition is satisfied, but suppose [$W > p \cdot y$ and $R(y, x) > \underline{R}(x, x)$ for some $y \in X$]. Since $W > p \cdot y$, by Assumption 6.21, there exists $z \in X$ such that

$$W \geqslant p \cdot z \tag{6.9}$$

and

$$R(z, y) \sim \bar{R}(x, x) \quad \text{and} \quad R(y, z) \sim \underline{R}(x, x). \tag{6.10}$$

Given $R(y, x) > \underline{R}(x, x)$, by Proposition 6.6, we have $R(z, x) \sim \bar{R}(x, x)$ and $R(x, z) \sim \underline{R}(x, x)$. Given (6.9) and [$x \in C(A(p, W))$], this contradicts Assumption 6.23. Hence, $p \cdot y \geqslant W$. Since $x \in C(A(p, W))$, clearly, $W \geqslant p \cdot x$. □

The following proposition follows as a corollary of Proposition 6.25.

Proposition 6.26. *Suppose the consumer satisfies Assumptions* 6.21 *and* 6.23. *Suppose for some price vector p and wealth W, and for some* $x \in X$, $A(p, W) \in K$ *and* $x \in C(A(p, W))$.

(1) *Let* (6.4) *hold so that for some* $x^*, y^* \in X$, *we have* $R(x^*, y^*) > \underline{R}(x^*, x^*)$) *and*

$$C(A') = H^1(A', R, x^*, y^*)$$

for all $A' \in K$. *Then* $p \cdot x \leqslant p \cdot y$ *for all* $y \in X$ *such that* $R(y, x) \geqslant R(x^*, y^*)$.

(2) *If* (6.5) *holds, then* $p \cdot x \leqslant p \cdot y$ *for all* $y \in X$ *such that* $R(y, x) \geqslant R(x, y)$.

(3) *If* (6.6) *holds, then* $p \cdot x \leqslant p \cdot y$ *for all* $y \in X$ *such that* $R(y, x) \geqslant m(\{x\} \times A)$.

Remark 6.27. Propositions 6.25 and 6.26 show that even when the consumer's preference ordering is soft, under suitable assumptions, preference maximization implies expenditure minimization in the appropriate sense.

Propositions 6.30 and 6.31 below explore the implication of expenditure minimization, in some appropriate sense, for preference maximization in the corresponding sense. We proceed to these two propositions via Assumption 6.28 and Lemma 6.29.

Assumption 6.28 (*Continuity*). For all $(x, y) \in X^2$,

$$\{(z, w) \in X^2 \mid R(z, w) \geqslant R(x, y)\}$$

and

$$\{(z, w) \in X^2 \mid R(x, y) \geqslant R(z, w)\}$$

are both closed in X^2.

Lemma 6.29. *Let the consumer satisfy Assumptions* 6.21 *and* 6.28. *Let p be a given price vector, and let* $x \in X$ *be such that for all* $y \in X$, *if* $p \cdot y < p \cdot x$ *then* $R(x, y) > \underline{R}(x, x)$. *Let there exist* $z \in X$ *such that* $p \cdot z < p \cdot x$. *Then* $R(x, y) \sim \bar{R}(x, x)$ *for all* $y \in A(p, p \cdot x)$.

Proof. Let the hypothesis of Lemma 6.29 be satisfied. We first prove that for all $y \in A(p, p \cdot x)$,

$$\text{if } p \cdot y < p \cdot x, \text{ then } R(x, y) \sim R(x, x). \tag{6.11}$$

If $y \in A(p, p \cdot x)$ and $p \cdot y < p \cdot x$, then there exists $e \in X$ such that $e > y$, and $p \cdot e < p \cdot x$. Since $e > y$, by Assumption 6.21, $R(e, y) \sim \bar{R}(x, x)$ and $R(y, e) \sim \underline{R}(x, x)$. However, given $p \cdot e < p \cdot x$, we must have $R(x, e) > \underline{R}(x, x)$. Given [$R(x, e) > \underline{R}(x, x)$ and $R(e, y) \sim \bar{R}(x, x)$ and $R(y, e) \sim \underline{R}(x, x)$], we have, by Proposition 6.6, $R(x, y) \sim \bar{R}(x, x)$. Noting that $R(x, x) \sim \bar{R}(x, x)$ (by the reflexivity of R) and that $\geqslant$ is an ordering over $X^2 \times \{R\}$, we have $R(x, y) \sim R(x, x)$.

Now we show that for all $y \in A(p, p \cdot x)$,

$$\text{if } p \cdot y = p \cdot x, \text{ then } R(x, y) \sim R(x, x). \tag{6.12}$$

Suppose $p \cdot y = p \cdot x$. Then consider z such as referred to in the hypothesis of Lemma 6.29. Consider any $y' \in [z, y[$. Since $p \cdot y = p \cdot x$ and $p \cdot z < p \cdot x$, we have $p \cdot y' < p \cdot x$ for all $y' \in [z, y[$, and hence, by (6.1), $R(x, y') \sim R(x, x)$. Hence, by Assumption 6.28, $R(x, y) \sim R(x, x)$.

Lemma 6.29 follows from (6.11), (6.12) and the reflexivity of R. □

Proposition 6.30. *Let the consumer satisfy Assumptions* 6.21 *and* 6.28. *Let* $x, x^*, x', y' \in X$

be such that $R(x', y') > \underline{R}(x', x')$, $R(x, x^*) \geqslant R(x', y')$, *and [for all* $y \in X$, *if* $R(y, x^*) \geqslant R(x', y')$, *then* $p \cdot x \leqslant p \cdot y$]. *Suppose there exists* $z \in X$, *such that* $p \cdot z < p \cdot x$. *Then*

$$x \in H^1(A(p, p \cdot x), R, x', y').$$

Proof. Let the hypothesis of Proposition 6.30 be satisfied. Since [for all $y \in X$, if $R(y, x^*) \geqslant R(x', y')$, then $p \cdot x \leqslant p \cdot y$], we have [for all $y \in X$, if $p \cdot y < p \cdot x$, then $R(x', y') > R(y, x^*)$], and hence [for all $y \in X$, if $p \cdot y < p \cdot x$, then $\bar{R}(y, y) > R(y, x^*)$]. By the connectedness of R, it follows that [for all $y \in X$, if $p \cdot y < p \cdot x$, then $R(x^*, y) > \underline{R}(y, y) \sim \underline{R}(x, x)$]. Noting $R(x, x^*) > \underline{R}(x', y) \sim \underline{R}(x, x)$, by max–min transitivity, we have $R(x, y) > \underline{R}(x, x)$ for all $y \in X$ such that $p \cdot y < p \cdot x$. By Lemma 6.29, it follows that for all $y \in A(p, p \cdot x)$, $R(x, y) \sim \bar{R}(x, x)$. Hence

$$x \in H^1(A(p, p \cdot x), R, x', y'). \qquad \square$$

Proposition 6.31. *Let the consumer satisfy Assumptions* 6.21 *and* 6.28. *Let* $x^* \in X$, *and let* $x \in X$ *be such that* $[R(x, x^*) \geqslant R(x^*, x)]$ *and [for all* $y \in X$, *if* $R(y, x^*) \geqslant R(x^*, y)$], *then* $p \cdot x \leqslant p \cdot y$]. *Let there exist* $z \in X$ *such that* $p \cdot z < p \cdot x$. *Then*

$$x \in H^2(A(p, p \cdot x), R).$$

Proof. Let the hypothesis of Proposition 6.31 be satisfied. Since $[R(x, x^*) \geqslant R(x^*, x)]$, by the connectedness of R, we have $R(x, x^*) \geqslant \underline{R}(x, x)$. Since [for all $y \in X$, if $R(y, x^*) > R(x^*, y)$], then $p \cdot x \leqslant p \cdot y$], it follows that for all $y \in X$, if $p \cdot y < p \cdot x$, then $R(x^*, y) > R(y, x^*)$. Hence, by the connectedness of R, for all $y \in X$, if $p \cdot y < p \cdot x$, then $R(x^*, y) > \underline{R}(x, x)$. Noting $[R(x, x^*) > \underline{R}(x, x)]$, by max–min transitivity of R, it follows that for all $y \in X$, if $p \cdot y < p \cdot x$, then $R(x, y) > \underline{R}(x, x)$. Then, by Lemma 6.29, it follows that $R(x, y) \sim \bar{R}(x, x)$ for all $y \in A(p, p \cdot x)$. Then clearly

$$x \in H^2(A(p, p \cdot x), R). \qquad \square$$

Remark 6.32. Proposition 6.30 and 6.31 show that, if the consumer satisfies either (6.4) or (6.5), then expenditure minimization in an appropriate sense implies preference maximization. However, we have not been able to formulate and prove a corresponding proposition for (6.6).

Our last result (Proposition 6.33) in this section shows that the 'substitution theorem' which is a basic result in the theory of consumers' behaviour based on crisp orderings, also holds in the more general framework of soft orderings.

Proposition 6.33. *Let the consumer satisfy Assumptions* 6.21 *and* 6.23. *Let* $x, x' \in X$, p, p', W *and* W' *be such that*

$$A(p, w) \in K, \qquad A(p', W') \in K,$$

$$x \in C(A(p, W)), \qquad x' \in C(A(p', W'))$$

and $W' = p' \cdot x$. *Then* $(p' - p) \cdot (x' - x) \leqslant 0$.

Proof. Since $x' \in C(A(p', W'))$ and $W' = p' \cdot x$, we have $p' \cdot x \geqslant p' \cdot x'$, and hence

$$p' \cdot (x' - x) \leqslant 0. \tag{6.13}$$

Since $x' \in C(A(p', W'))$ and $x \in A(p', w')$, by Assumption 6.23 and connectedness of R, it follows that $R(x', x) > \underline{R}(x, x)$. Hence, given $x \in C(A(p, W))$, by Proposition 6.25, $p \cdot x' \geqslant p \cdot x$. Therefore

$$-p \cdot (x' - x) \leqslant 0. \tag{6.14}$$

From (6.13) and (6.14) it follows that

$$(p' - p) \cdot (x' - x) \leqslant 0. \qquad \square$$

7. Concluding remarks

In this paper we have introduced the notion of soft sets, which constitutes an ordinal formulation of the notion of vague sets, as distiguished from the cardinal formulation in the traditional theory of fuzzy sets. We have explored the relation between our notion of soft sets and Goguen's [8] notion of L-sets. It has been shown that a close logical relation exists between the two concepts. We have also considered several applications of soft binary relations to the theory of vague preferences and rational choice in general and also to the specific, economic problem of consumers' behaviours in the presence of vague preferences. It is our belief that, in the context of many problems of interest to social scientists, the ordinal formulation provides an intuitively more satisfactory method of modelling vagueness than the traditional cardinal formulation. It is also our belief that the ordinal version has many potential applications in the social sciences, especially in economics. The results in Section 6 of our paper constitute only a few examples of such applications.

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