The emergence of non-Walrasian general equilibrium theory has played a critical role in providing microfoundations for macroeconomics. This is especially true of fix-price general equilibrium theory and post-Keynesian macroeconomics. The purpose of the present paper is to develop a systematic geometry for non-Walrasian equilibrium analysis, including the theory of coupons equilibrium. This should provide a useful analytical and pedagogic tool for studying the microeconomic base of macro issues like unemployment and repressed inflation.

1. Introduction

The emergence of non-Walrasian general equilibrium theory and its rapid advance through the seventies have played a critical role in providing a common analytical platform for micro- and macroeconomic theory. Despite this important role, the dissemination of non-Walrasian ideas has been quite slow. The reason for this is the terseness of the subject which renders its main theorems inaccessible, not only to the laity but to much of the economics profession.

The idea of a Walrasian equilibrium can be transmitted to the virtual layman with the help of the Edgeworth-Bowley box. It would be useful to have a systematic geometry for non-Walrasian equilibrium analysis as well. Some geometric developments have indeed occurred in this area: Grandmont (1977a), Bohm and Muller (1977) and Dreze and Muller (1980). The aim of the present paper is to push this line of inquiry towards completion. By drawing together the scattered developments in the literature, this paper provides a geometry which can capture the problems and theorems of fix-price general equilibrium theory of the kind developed in Dreze (1975),
An important focus of the paper is on the "coupons equilibrium" and its variants. But, since a purpose of the paper is pedagogic, it first goes through discussions of existence, uniqueness, and optimality before taking on the subject of coupons more extensively.

Throughout the paper I consider an economy with two individuals and three goods—one of which is money. An Edgeworth box for such an economy is indeed really a box, in the sense of being three dimensional, as shown in Figure 1. However, for a clear geometric analysis we need to work in two dimensions, so our first step is to translate our domain of analysis from three to two dimensions.

Note that, since in a fix-price model the initial endowments of the two agents, a and b, and prices are fixed, we can focus attention entirely on the budget hyperplane inside the Edgeworth box. This is shown as ABCDE in Figure 1. The strategy is to take out the slab ABCDE, lay it out flat and do the analysis on it.

2. Preliminary Geometric Concepts

Throughout the paper I consider three goods, labeled 0, 1, and 2, of which good 0 is supposed to be money. The prices of the three goods are given by $(p_0, p_1, p_2) = p > 0$. There are two individuals labeled a and b. The endowment of individual i is given by $(w^i_0, w^i_1, w^i_2) = w^i$. We shall use w to denote $(w^a, w^b)$.

The first problem that we confront is this. If we slice the Edgeworth box of this economy with a budget hyperplane, the area of the plane which lies within the box is our relevant domain. I shall call such an area the PE-domain since it is fixed as soon as we specify prices and endowments. What does a PE-domain look like?

It could look like ABCDE in Figure 1 but it could also look very different. It all depends on where the plane slices through the Edgeworth box. If it slices close to an origin it could be a triangle like FGH in Figure 1. But it could also be a parallelogram or a polygon.

Fortunately, there is an easy way of characterizing PE-domains in general. In order to do so, let us follow the convention...
of describing two triangles on a 2-dimensional Euclidean surface as parallel if, for each side on each triangle, the opposite side on the other is parallel. That is, from each triangle you get the other by turning it around 180° and miniaturizing or expanding it if necessary. Figure 2 gives two examples of parallel triangles.

It is easy to see that a PE-domain takes those shapes that can be generated by the intersection of two parallel triangles. Thus a PE-domain could be any of the following two areas in Figure 2: ABCDE and JKLM. One can sharpen ones intuition of why this should be so by cutting through loaves of bread in different directions and then contemplating the cross-sections.

To understand this characterization more formally it is useful to begin by considering a 2-by-2 Edgeworth box. Given an endowment and a price vector we get a PE-domain like AB in Figure 3. But there is another way of viewing AB. Given the price and endowment we can first draw agent a's budget set, CB. Similarly we can draw b's budget set, AD. The PE-domain is nothing but the intersection of these two budget sets.

My characterization in the 3-dimensional case uses exactly this principle. The two parallel triangles are the budget sets of the two players. That is, ABCDE in Figure 1 could be derived by extending this to the axes of player a to form a's budget set, doing the same for b and then taking the intersection of the two budget sets.

We shall follow some conventions throughout the paper. Player a's budget set will be drawn vertically with the top corner representing good 0 (that is, money) and the right-hand bottom corner, good 1. This is shown in Figure 4. The budget triangle set of player

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5I am using the term “budget set” to refer to the northeastern surface of the feasible set defined by the budget constraint and non-negativity constraints.
Figure 2.

$b$ will therefore be upside down. The PE-domain will be marked by an unbroken line as shown in Figure 4. The endowment point in the PE-domain will be marked \( w \).

Now any point in an agent's budget set is a vector of commodities. Consider \( x \) and \( y \) in Figure 4. Which one of these two points contains a higher amount of, say, good 1 for person \( a \)? To answer this we have to draw straight lines through \( x \) and \( y \) which are parallel to the line \( 02 \). Whichever of \( x \) and \( y \) is on the line closer to point 1 has a larger amount of good 1. In the figure, \( y \) has more of good 1 for person \( a \). Clearly for person \( b \), \( x \) has more of good 1. Hence the broken lines through \( x \) and \( y \) could be described as \textit{iso-good 1 lines}.

The next question is concerning indifference curves in the PE-domain. I shall make the usual assumptions on a consumer's preference relation (that is, it must (i) be an ordering, (ii) subsume vector-dominance, (iii) be strictly convex and (iv) be continuous). In the 3-dimensional Edgeworth box, indifference surfaces are saucer shaped. The intersection of these with the PE-domain gives the

Figure 3.
indifference curves in the PE-domain. These will be strictly convex\(^4\) and have a bliss point. We shall denote a’s and b’s bliss points by \(B_a\) and \(B_b\). In the context of the full Edgeworth box, these are, of course, constrained bliss points, representing each player’s highest-utility point on his budget set. Thus in Figure 4, \(B_a\) is the point which gives the greatest utility among all points in the triangle 012.

Pareto optima can be defined in many different ways in fix-price general equilibrium theory. We could think of a constrained Pareto optimum as a point on the PE-domain such that no other point on the PE-domain would constitute a Pareto improvement. In the terminology of Dreze and Muller (1980) such a point would be Pareto optimal at the “second level.” The collection of all constrained Pareto optima will be referred to as the constrained contract curve. The line joining \(B_a\) and \(B_b\) and going through all points of tangency between indifference curves constitutes our contract curve as shown in Figure 4.

\(^4\)This follows from my assumption of strict convexity of preference. Relaxing the “strict” part of the assumption can complicate the analysis considerably (see Grandmont 1977b).
3. Rations and Consumer Choice

Rationing in standard non-Walrasian economics takes the form of upper limits on net supply and net demand of an agent or a consumer, imposed separately on each good. I shall go along with this here and assume that a consumer can never be forced to trade more than he wants but he can be forced to trade less. This is known as the “no forced trading” assumption in the literature.\(^5\) We shall also assume, as is common in this literature, that money is never rationed.

A ration on net trade in commodity 1 is represented by a straight line parallel to the line 02 in Figure 4 drawn in the PE-domain. A consumer faced with such a ration is free to choose any consumption point on the PE-domain on the side of the ration line on which \(w\) is located. This follows from the no-forced-trade assumption. Hence whenever we draw a ration line, if this line does not pass through \(w\), there is no need to indicate which is the feasible side of the ration line for it must always be the side on which \(w\) happens to be. If the ration line passes through \(w\) it will be necessary to indicate which side is feasible.

Consider person \(a\) in Figure 5. If he faces a ration on good 1 represented by line \(AB\), he would choose to consume at \(E\). Note that this ration is a restriction on the net supply of good 1. If a ration on good 1 is imposed at \(A''B''\) (that is, a ration on demand) it is easy to see that the agent would choose to consume at \(B_a\). It is therefore clear that by using a ration only on good 1, consumer \(a\) can be made to choose any point on the line \(B_aE'\), which is derived by marking off the tangencies of lines parallel to 02 and the agent’s indifference curves.

Moreover, the consumer cannot be directed to any other point by rationing only good 1. The line \(B_aE'\) will be referred to in this paper as agent \(a\)’s 1-rationed locus.

We could use a similar exercise and derive the agent’s 2-rationed locus, that is, the locus of points which the agent could be made to choose by rationing good 2 only. This is shown as line \(B_aD\).

It is interesting to note here that a point like \(E\) (where an indifference curve is tangential to line \(AB\)) could represent a lower

\(^5\)It may be noted that there are other ways of defining “no-forced trading.” For instance, “no-forced trading” may be defined as any quantity restriction on an agent, \(i\), which does not take away his right to consume his endowment, \(w'\), if he so wishes.
consumption of good 2 than point \( B_a \). That is, a supply restriction on good 1 could enhance \( a \)'s supply of good 2.\(^6\) This is known as the "spillover effect" (Benassy 1982) or "forced substitution" (Kornai 1982), though Kornai speaks about this in the context of excess demand instead of supply.\(^7\)

Before we can do a general equilibrium analysis we must go a step further and locate all points which an agent can be made to choose by using quantity rations on only goods 1 and 2. In Figure 5 this is represented by all points in the area \( B_a E'wD \), that is, all points enclosed by the 1-rationed locus, 2-rationed locus, and lines through \( w \) which are parallel to \( 01 \) and \( 02 \).

This is easy to see. Consider a point like \( H \). Suppose first there is a ration on good 1 represented by line \( AB \). Now place a ration on good 2 by a line which goes through \( H \). Faced with these two rations the agent will find it optimal to choose point \( H \). This follows from the convexity of preferences. On line \( AB \) point \( E \) is the best. If he is prevented from reaching \( E \), he would move as close to \( E \) as is possible. This is given by point \( H \).

\(^6\)It could also transform an agent from a net supplier of good 2 to a net demander of good 2. That is, it is possible for the agent's 1-rationed locus to go from \( B_a \) to some point on the line segment \( A'w \). I will not consider such a case here explicitly because it does not change the results, and the little additional geometric complication is easy to handle.

\(^7\)It is also arguable that Kornai's "forced substitution" is a more complicated concept. For one, Kornai allows for the fact that in the event of being rationed in a market, a consumer will not only turn to other goods but may take to queuing or postponement of purchase. Moreover, if a shortage is chronic, consumers may adjust so well to the situation that there may be few overt behavioral manifestations of the shortage.
The collection of all points which an agent can be made to choose by using quantity rations on goods 1 and 2 will be referred to as the 1–2 rationed zone. As already pointed out, in Figure 5, this is given by $B_sDwE'$.

4. General Equilibrium

Now we are in a position to give a full geometric representation of general equilibrium with quantity rationing à la Dreze (1975). I do this in two steps. A point, $e$, in the PE-domain and quantity rations on goods 1 and 2 on agents $a$ and $b$ constitute a quasi-equilibrium if both agents choose to consume at $e$ when faced with the given quantity rations. A quasi-equilibrium is an equilibrium (or Dreze equilibrium) if there does not exist any good for which one agent is demand rationed and the other is supply rationed.\(^5,6\)

The existence of a quasi-equilibrium in general is easy to see. The original endowment point, $w$, with complete quantity restrictions (in the sense of zero trade being allowed) is clearly a quasi-equilibrium. This is not a very interesting equilibrium, and the purpose of Dreze's additional restriction is to rule out the no-trade point from qualifying as an equilibrium wherever possible.

The formal proof of the existence of an equilibrium is quite terse. One advantage of the diagrammatic method developed above is that one can "see" and get an intuitive idea of this theorem. For this, it is useful to first identify the set of all quasi-equilibria; and then search for the equilibrium within this set.

In Figure 6, draw lines parallel to $01$ and $02$ through the initial endowment point $w$. This partitions the PE-domain into four zones which I shall refer to as the north, east, south and west zones.

\(^5\)Dreze's (1975) original definition requires also that for each good the quantity ration imposed on the individuals' net trade is of a fixed magnitude, quite irrespective of who the individual happens to be. The slight generalization that I present here has been discussed in the formal literature (see, for example, Grandmont 1977a, 1977b).

\(^6\)While this requirement (that both supply and demand for the same good must not be rationed at the same time) seems at first sight unquestionable, the "Hungarian school" reminds us that this may indeed be a strong assumption. This is yet another point of divergence between the Hungarian model of shortage and the literature I am discussing. As Kornai (1982, p. 38) notes, "It frequently happens at the microlevel that neither the collectivity of sellers nor that of buyers can perfectly fulfill their intentions." Turning to our framework, once we bring this kind of an argument in, it is possible that the "equilibrium" will settle in the interior of the set of quasi-equilibrium points.
Now the constrained bliss points of the two agents could lie in the (i) same zone, (ii) adjacent zones, or (iii) opposite zones.

It is convenient to analyze the three cases separately. Consider (i) first, which is illustrated in Figure 6. Let $B_aA$ and $B_aF$ ($B_bC$ and $B_bD$) be agent $a$'s (agent $b$'s) 2-rationed locus and 1-rationed locus. From the discussion in the previous section it is obvious that for any point in the area $wCeF$, it is possible to impose quantity rations on the two agents such that the constrained optimum of both agents happen to occur at that point. Hence all points in $wCeF$ are quasi-equilibria.

Now consider any point in this area apart from $e$. Take, for instance, point $m$. Note that both agents have to be quantity constrained for good 1 and while agent $a$'s supply is restricted, for agent $b$ it is demand that is restricted. (Remember that agent $b$'s budget set has to be viewed from the opposite side from that of $a$'s). Hence $m$ cannot be an equilibrium. It is easy to check that point $e$ is an equilibrium.

Using a similar analysis the other cases are simple to figure out. Hence I simply illustrate these in Figure 7, in two self-evident diagrams. In the left-hand panel, the line segment $ew$ represents quasi-equilibria, and $e$ is an equilibrium. In the right-hand panel there is a unique quasi-equilibrium and equilibrium. This is the zero trade point, $w$.

In the above analysis I have omitted the case discussed in footnote 5, where the imposition of a quantity restriction on one commodity changes an agent from a net supplier (demander) to a net demander (supplier). This is because the complication is easy to handle and leaves the existence proposition unchanged.
From the three cases illustrated in Figures 6 and 7 it appears that the Dreze equilibrium is unique. However, the appearance is false. I shall now illustrate using the geometric technique developed thus far that an economy may have several Dreze equilibria, and this is so even when individual preferences are strictly convex.

In Figure 8 let player b's 2-rationed locus be the line $B_bC$. It is possible for this to overlap with player a's 1-rationed locus, which is represented by $B_aF$. Let $MN$ be the segment over which the two loci overlap. It may now be checked that every point on $MN$ is an equilibrium.

Consider a point, $e$, on the $MN$ segment and draw two lines through it—one parallel to $01$ and the other parallel to $02$. Let the former represent a ration on player b and the latter a ration on player a. It is clear that with these constraints both players would choose to consume at point $e$. Hence $e$ is a quasi-equilibrium. Since this quasi-equilibrium is sustained by rationing a's supply of good
and b's supply of good 2, point e is also an equilibrium. Since e was an arbitrary point on MN, this establishes the non-uniqueness of the Dreze equilibrium. It is easy to check that the set of Dreze equilibria may be neither unique nor a continuum; and may be finite.

5. Optimality

This section on optimality is kept brief because it overlaps with Bohm and Muller (1977). It is not omitted altogether for the sake of completeness and because the next section requires a familiarity with the matter covered here.

A Pareto optimal point is one such that there is no other point in the Edgeworth box at which both agents are better off. A constrained Pareto optimal point, on the other hand, is defined with respect to the PE-domain instead of the Edgeworth box. A formal definition is given in Section 2.

Consider a fix-price economy with endowment w and price vector p. Suppose further that p is not a Walras equilibrium price. Consider now the (Dreze) equilibrium of this economy. Is this Pareto optimal? The answer is: It may be so but only in very special cases. This is illustrated in Dreze and Muller's paper, in particular their Figure 1.

I shall concentrate here on the relation between equilibria and constrained Pareto optima. It will first be shown that an equilibrium of the kind discussed in the previous section need not in general be a constrained Pareto optimum.

Consider, for instance, case (ii) of the previous section. An equilibrium for this case is illustrated in the left panel of Figure 7. From the definition of an i-rationed locus it follows that b's indifference curve through h and a's indifference curve through e will both be tangential to the same line as shown in the figure. Hence the constrained contract curve must pass through the segment eh but not through e or h. It follows that the equilibrium which is at point e is not a constrained Pareto optimum.

Before turning to the analysis of coupons equilibria, I want to comment on a theorem which is the counterpart of the so-called second basic theorem of welfare economics in fix-price economies. This states that every constrained Pareto optimum point could be sustained as an equilibrium if we were free to alter the initial endowment within the PE domain.

To see this, first note that if the constrained contract curve
passes through \( w \), then \( w \) is a Dreze equilibrium. It immediately follows that if in a fix-price economy a point \( e \) is a constrained Pareto optimum, then if we change the initial endowment to \( e \), \( e \) would be an equilibrium.

6. Coupons Equilibria

A very interesting and alternative method of rationing is via the use of coupons. The advantage of this method is that it can achieve constrained Pareto optimality (Dreze and Muller 1980; Hahn 1978). The best way to understand a coupons economy is to superimpose it on the kind of economy discussed thus far. So suppose now, over and above the existing endowments, each of the two individuals are handed over \( c \) coupons.\(^{10}\) In addition coupon prices are announced for each of the three goods. Let \( q_i \) be the coupon price of good \( i \). These can be positive, negative or zero. A coupon price represents the price of a good in coupons. Thus, if an agent buys one unit of good \( i \), he has to pay \( q_i \) coupons. This is in addition to the money prices. Hence, from now on, to buy a good an agent has to pay in two currencies: money and coupons. Similarly, when supplying goods, he receives both currencies. If, for instance, agent \( i \) sells off all his endowment of goods 1 and 2, he receives \( w_i^1 q_1 + w_i^2 q_2 \) coupons and \( w_i^1 p_1 + w_i^2 p_2 \) money. It will also be assumed throughout that the coupon price of money is zero. That is, \( q_0 = 0 \).

It follows that in an economy with endowment fixed at \( w \) and price at \( p \), if each individual is given an additional endowment of \( c \) coupons and \( q = (q_0, q_1, q_2) \) is the coupon price of goods, a consumer \( i \) can choose any consumption vector \( (x_0^i, x_1^i, x_2^i) \), which satisfies the following two conditions:

\[
\begin{align*}
p_0 x_0^i + p_1 x_1^i + p_2 x_2^i &= p_0 w_0^i + p_1 w_1^i + p_2 w_2^i ; \\
q_1 x_1^i + q_2 x_2^i &\leq c + q_1 w_1^i + q_2 w_2^i .
\end{align*}
\]

In terms of geometry, let 012 in Figure 9 be the normal budget set given by (1), which we have been using all this time. What

\(^{10}\)Note that each agent’s coupon endowment is the same. In Dreze and Muller’s terminology the equilibrium that I define below is therefore a uniform coupons equilibrium. I make this assumption only for simplicity. The coupons economy described by Hahn has \( c = 0 \).
constraint (2) does is draw a straight line, like $AB$, across this space, such that the area that satisfies both (1) and (2) is given by $OBA2$. By varying $(c, q)$, the line $AB$ could be made to fall anywhere. This will be called the *coupon line*. Note that, as long as $c > 0$, the feasible side of the coupon line is always the one in which the endowment point $w$ is located. If $c = 0$ (the Hahn special case), either side could be feasible, depending on the values of $q_1$ and $q_2$. We shall put an arrow on $AB$ to mark the feasible side as shown in Figure 9. When we consider both players $a$ and $b$, as in the PE-domain, we shall use a broken line marked $A \hat{b} B$ to denote $b$'s coupon's constraint. The feasible side of $b$ will obviously be in the opposite direction from that of $a$. Hence we follow the convention of using the arrow to mark $a$'s feasible side.

Following Dreze and Muller, for an economy described by fixed endowment and price vectors, we define a *coupon equilibrium* as a point $e$ in the PE-domain, a coupons endowment, $c$, and a coupons price vector, $q$, such that, given the additional restriction imposed by $c$ and $q$, each agent chooses to consume at point $e$. A coupons equilibrium is described as Hahn-type if $c = 0$.

Figures 10 and 11 illustrate two alternative coupons equilibria. In Figure 11 we have a Hahn-type coupons equilibrium at $e$. If $A B$ is the coupon line, it is clear that $a$'s best consumption point that is feasible is given by $e$. This is also $B$'s chosen point. Hence this is a coupons equilibrium. Since this coupon line $A B$ (or $A \hat{b} B$) goes through $w$, $c = 0$, and the equilibrium is Hahn-type.

This figure makes intuitively clear what is indeed formally true, that a Hahn-type coupons equilibrium is always a constrained Pareto optimum. This is because a Hahn-type coupons equilibrium in a three-good economy is similar to a Walrasian equilibrium in a two-good economy.
While this is an encouraging result, as Dreze and Muller point out, an economy does not always have a Hahn-type coupons equilibrium. This is very easy to show with our geometric method.

Let us assume that the PE-domain and the agents’ indifference curves are such that, in the case illustrated in Figure 10, the constrained contract curve is a straight line. But suppose that the initial endowment point, instead of being where it is in Figure 10, happens to be on the straight-line extension of the contract curve as shown in Figure 11. It is immediately obvious that it is not possible to draw a straight line through $w$ such that both agents treat-
ing this as the coupon line will choose the same consumption point. In brief, a Hahn-type equilibrium does not exist.

Nevertheless a coupons equilibrium does exist. To see this, consider player b's constrained bliss point, $B_b$. Through this point draw a straight line which is tangential to $a$'s indifference curve which goes through $B_b$. Let this line be marked $AB$. Let $AB$ be treated as agent $a$'s coupon line. This automatically defines player $B$'s coupon line. This is shown as $\hat{A} \hat{B}$. Given these coupon lines, agent $a$ would choose to consume at $B_b$ as well. Hence we have described a coupons equilibrium. From the way this proof is constructed, it seems clear that a coupons equilibrium will always exist. This, as we know from Dreze and Muller's formal derivation, is indeed true.

The geometry also suggests that a coupons equilibrium is always a constrained Pareto optimum. This is also formally true, though it does hinge on the assumption of strict convexity of preferences (which has been assumed to be true throughout this paper).

It is interesting to note that the above analysis also suggests that if the constrained bliss points of the two players do not coincide (that is, $B_a \neq B_b$) and $(e, c, q)$ is a coupons equilibrium with $c > 0$, then, at equilibrium, one of the two players must be at his constrained bliss point.

7. Conclusion

The purpose of this paper was to develop a geometry for interpreting non-Walrasian general equilibrium theory. This could be especially valuable because of the critical role that non-Walrasian economics has played in providing micro-foundations for macroeconomics. It is hoped that the geometry developed here will be of more than just heuristic value and will provoke new research in the same way that the Edgeworth box has stimulated ideas within the domain of Walrasian economics.

There are several directions that one may pursue from here. Note, for instance, that there must exist at least one point on the constrained contract curve which is not just constrained Pareto optimal, but Pareto optimal. A question deserving future investigation is whether we can devise reasonable rationing schemes such that the equilibrium will be driven to such an optimal point. A priori it seems that two-part tariffs or nonlinear pricing can play a role in devising such schemes. Other possible directions of research in-
clude the introduction of production and comparative statics to evaluate the effects of changes in exogenous variables and policy parameters.

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References
