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# A characterization of the class of rationalizable equilibria of oligopoly games

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The set of rationalizable strategies of a firm in a quantity-setting oligopoly is characterized. It is shown that as the number of firms in the industry increases, the rationalizable solution converges to the interval from zero to the monopoly output.

## 1. Introduction

This paper characterizes the set of rationalizable strategies for quantity-setting oligopolies and examines how this set varies with industry size. It is shown that when industry size is small, the rationalizable solution is no different from the Cournot outcome. But as the size of the industry increases the set of rationalizable outcomes grows and converges to the interval from zero to the monopoly output. The paper draws on Bernheim (1984), which characterizes the rationalizable solution in the linear special case, and some literature on stability in oligopoly [e.g., Theocharis (1960), Friedman (1977) Gabay and Moulin (1980)].

## 2. An oligopoly game

Consider an industry with $n$ firms, where $n \geq 2$. The firms are identical in the sense of having identical cost function. Firm $i$'s *profit function* is a continuous mapping,

$$\pi : \mathbb{R}_+^2 \to \mathbb{R}_+,$$

such that if the $n$ firms produce $(x_1, \ldots, x_n) \in \mathbb{R}_+^n$, then $i$'s profit is given by $\pi(x_i, \Sigma_{j \neq i} x_j)$.

Given the profit function, firm $i$'s *reaction function* is a mapping,

$$\phi : \mathbb{R}_+ \to \mathbb{R}_+,$$

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such that given $r \in \mathbb{R}_+$, $\phi(r) = \text{argmax}_x\ \pi(x, r)$. That is, given that the rest of the firms supply $r$ units, firm $i$'s best response is to produce $\phi(r)$. It is being assumed that, for all $r$, $\phi(r)$ is unique, that is, $\phi$ is indeed a function.

It will be assumed in this paper that

(i) $\phi$ is continuous,

(ii) there exists an output level, $x^c$, such that, for all $r \geq x^c$, $\phi(r) = 0$ and if $r < r' \leq x^c$, then $\phi(r) > \phi(r')$, and

(iii) $x^c > \phi(0)$. We denote $\phi(0)$ by $x^M$.

It should be obvious that $x^M$ is an industry's monopoly output; and it will be seen that, in some situations, $x^c$ is a competitive industry's output.

Given assumptions (i)–(iii), no generality is lost if we assume that there exists a large enough number $z$ such that each firm $i$'s strategy, $x_i$, is confined to the interval $[0, z]$. I shall, in particular, fix a $z \geq x^c$. Given a $\pi(\cdot)$, such that (i)–(iii) are satisfied, and each firm selects an output from $[0, z]$, we have a well-defined normal-form game. Such a game will be called an *oligopoly game*. Since the profit function plays no direct role, it seems reasonable to treat the reaction function, $\phi$, as a primitive and to treat each $\phi$ as an oligopoly game.

An output vector or strategy combination, $(x_1^*, \ldots, x_n^*)$ is a *Cournot equilibrium* if, for all $i$, $x_i^* = \phi(\sum_{j \neq i} x_j^*)$.

Though the games we are considering, with their infinite strategy sets, belong to the class of games discussed by Bernheim (1984) I shall define rationalizability by the method used by Pearce (1984) and, in addition, attention is confined to pointwise conjectures.

For each firm $i$, define a sequence of sets $\{H_i(t)\}$, where $t = 0, 1, 2, \ldots,$ as follows:

$$H_i(0) = [0, z],$$

And, by induction, for all $t \geq 1$,

$$H_i(t) = \left\{ x_i \in H_i(t-1) \mid \text{there exists } y_j \in H_j(t-1), \text{ for all } j \neq i, \text{ such that } x_i = \phi\left( \sum_{j \neq i} y \right) \right\}.$$

The set of *rationalizable strategies* of player $i$, $R_i$, is defined as

$$R_i = \bigcap_{t=0}^{\infty} H_i(t).$$

If $x^* \in R_1 \times \ldots \times R_n$, $x^*$ will be referred to as a *rationalizable solution* or *equilibrium*.

It is easy to verify that every oligopoly game has a rationalizable solution [Bernheim (1984)]. Since firms are identical, $R_i = R_j$, for all $i, j$. Hence we shall usually drop the firm subscript and write $R$ for $R_i$.

## 3. Rationalizability and industry size

In order to characterize the set of rationalizable equilibria of an oligopoly game, it is useful to define a mapping

$$\psi : [0, z] \to [0, z],$$

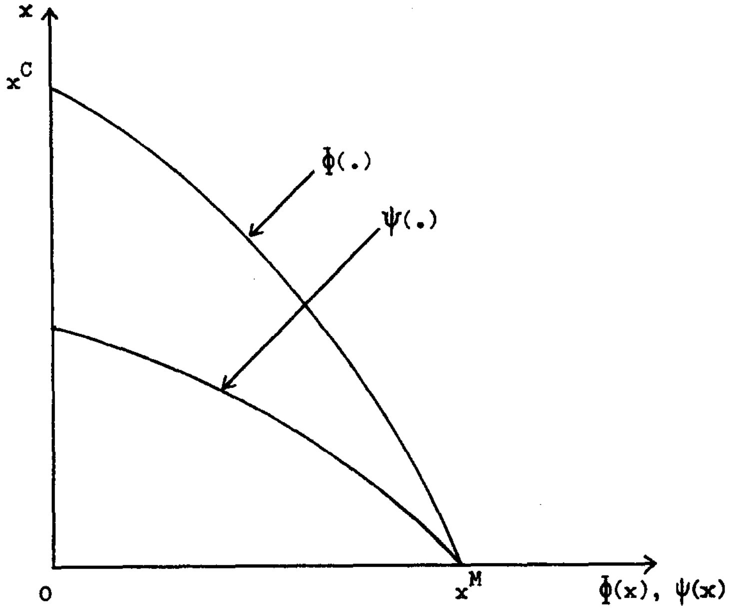


Fig. 1.

such that, for all $x \in [0, z]$, $\psi(x) = \phi((n-1)x)$. It is easy to check that $\psi([0, z]) \subset [0, z]$: Clearly $\psi(0) = x^{M} \in [0, z]$, since $x^{M} \leq x^{c} \leq z$. Note that $\psi(z) = \phi((n-1)z) = 0$, since $(n-1)z \geq x^{c}$. Since $\psi$ is (inversely) monotonic [see (ii)], it follows that $\psi([0, z]) \subset [0, z]$.

Fig. 1 depicts a reaction function, $\phi$, and the function, $\psi$, for $n = 3$. It is clear that the inverse of the $\phi(\cdot)$ function is $(n-1)$-times as high as the inverse of the $\psi(\cdot)$ function.

Note that for every oligopoly with $n$ firms, we have a unique $\psi$, where $\psi$ depends on $n$. Consider now a duopoly consisting of two identical firms with reaction functions given by $\psi$. For every oligopoly, we shall refer to the duopoly constructed in this manner as its *associated duopoly*. What will now be shown is that to find the rationalizable strategies of any oligopoly, all we need to do is to locate the set of Nash equilibria of its associated duopoly and consider its rectangular hull.

Define a set, $D$, as follows:

$$D = \{ x \in [0, z] \mid x = \psi(\psi(x)) \}.$$

*Lemma 1.* (*a*) *There exists* $\underline{x} \equiv \min D$ *and there exists* $\bar{x} \equiv \max D$.
(*b*) $\underline{x} = \psi(\bar{x})$ *and* $\bar{x} = \psi(\underline{x})$.

*Proof.* (a) Since $\phi$ is continuous, $\psi(\psi(\cdot))$ must be continuous. Since $[0, z]$ is compact and convex, by Brouwer's fixed point theorem, $D$ is non-empty. Define $g(x) = \psi(\psi(x)) - x$, for all $x$. Clearly, $D = g^{-1}(0)$. Since $g$ is continuous and $\{0\}$ is compact, $D$ is compact and hence (a) must be true. (b) Note that if $x \in D$, then $\psi(x) \in D$. Suppose $\psi(\bar{x}) < \underline{x}$. Then $\psi(\bar{x})$ is not an element of $D$, which is a contradiction. Next, suppose $\psi(\bar{x}) > \underline{x}$. Then, by (ii) $\bar{x} = \psi(\psi(\bar{x})) < \psi(\underline{x})$. Since $\psi(\underline{x}) \in D$, this is a contradiction. Hence $\underline{x} = \psi(\bar{x})$. This, in turn, implies that $\psi(\underline{x}) = \psi(\psi(\bar{x})) = \bar{x}$. Q.E.D.

Using the definition of $\underline{x}$ and $\bar{x}$ introduced in the statement of Lemma 1, we can now characterize the set of rationalizable strategies of a player.

*Theorem 1.* *For every oligopoly game* $[\underline{x}, \bar{x}] = R$.

*Proof.* First note that

$$x \in [\underline{x}, \bar{x}] \to \psi(x) \in [\underline{x}, \bar{x}]. \tag{1}$$

Let $\underline{x} \le x \le \bar{x}$.

$$\to \psi(\bar{x}) \le \psi(x) \le \psi(\underline{x}), \text{ since } \psi \text{ is inversely monotonic.}$$

$$\to \psi(x) \in [\underline{x}, \bar{x}], \text{ by Lemma 1.}$$

Clearly $[\underline{x}, \bar{x}] \subset H_i(0)$, for all $i$. Suppose $[\underline{x}, \bar{x}] \subset H_i(t-1)$, for all $i$. Let $x \in [\underline{x}, \bar{x}]$. By (1), $\psi(x) \in [\underline{x}, \bar{x}]$. Since $x \in H_i(t-1)$, and $\psi(x) \in H_j(t-1)$, for all $j$, and $x = \psi(\psi(x)) = \phi((n-1)\psi(x))$, it follows from the definition of $H_i(t)$ that $x \in H_i(t)$. Hence, by induction, $x \in R$. Thus $[\underline{x}, \bar{x}] \subset R$.

The proof is completed by showing that $R \subset [\underline{x}, \bar{x}]$. Without loss of generality, suppose there exists $x \in R$ such that $x > \bar{x}$. Since $R$ is compact [Basu and Weibull (1991)], max $R$ exists. Let $x^* = \max R$. Since $x \le x^*$, $x^* \notin [\underline{x}, \bar{x}]$.

Let $x_* = \psi(x^*)$. Hence $x_* \in R$. Moreover, since $\psi$ is inversely monotonic, $x^* = \min R_i$. By the same argument, $x^* = \psi(x^*)$. Hence $\psi(\psi(x^*)) = x^* \in D$, which implies $x^* \le \bar{x}$. This is a contradiction. Q.E.D.

Thus far industry size has been held constant at $n$. Now the relation between rationalizability and industry size will be investigated. Note that $\underline{x}$, $\bar{x}$, $R$, $D$ and $\psi$ depend on $n$. This fact was not important till now since $n$ was fixed. Since $n$ will now be varied, I shall from now on write $\underline{x}(n)$, $\bar{x}(n)$, $R(n)$, $D(n)$ and $\psi_n$ for $\underline{x}$, $\bar{x}$, $R$, $D$ and $\psi$.

As the industry size, $n$, increases, it seems natural to expect coordination problems to grow and for a wider range of behavior to become justifiable by using a wider range of conjectures. The next theorem asserts that this is indeed so.

*Theorem 2.* *For every oligopoly game there exists an integer* $\bar{n}$ *such that for all* $n \ge \bar{n}$, $R(n) = [0, x^{\mathrm{M}}]$.

*Proof.* Consider an oligopoly game described by $\phi$. Let $\bar{n}$ be the smallest $n$ for which $x^{\mathrm{M}} \ge x^{\mathrm{c}}/(n-1)$. Hence $\bar{n}$ is any integer with the following properties:

$$(\bar{n}-1)x^{\mathrm{M}} \ge x^{\mathrm{c}} \tag{2}$$

and

$$(\bar{n}-2)x^{\mathrm{M}} < x^{\mathrm{c}}; \tag{3}$$

$$(2) \to \phi((\bar{n}-1)x^{\mathrm{M}}) \le \phi(x^{\mathrm{c}}) = 0,$$

$$\to \psi_{\bar{n}}(x^{\mathrm{M}}) = 0; \tag{4}$$

$$(3) \to \phi((\bar{n}-2)x^{\mathrm{M}}) > \phi(x^{\mathrm{c}}),$$

$$\to \psi_{\bar{n}-1}(x^{\mathrm{M}}) > 0. \tag{5}$$

Since $\psi_{\bar{n}}(D) = x^M$, (4) implies $x^M \in D_{\bar{n}}$ and $0 \in D_{\bar{n}}$. By Theorem 1, $[0, x^M] \subset R(\bar{n})$. It is obvious that $R(\bar{n}) \subset [0, x^M]$. Hence, $R(\bar{n}) = [0, x^M]$. Q.E.D.

Define $n^* = 1 + n^c/x^M$. From the proof of Theorem 2 it follows that $\bar{n}$ (the critical industry size where the rationalizable set expands to $[0, x^M]$) is the smallest integer greater than or equal to $n^*$. Next note that for a large class of oligopoly games $x^c$ has a neat interpretation. Consider the class where the cost function is linear and the demand curve is downward sloping. In such cases, $x^c$ happens to be the output that would be produced under perfect competition. Hence, the following theorem is obvious.

*Theorem 3. If the number of firms in an industry is at least as large as one plus the ratio of the output produced by the industry if it was perfectly competitive and the output produced by it if it was a monopoly, then and only then the set of rationalizable strategies of a firm consists of all production levels from* 0 *to the monopoly output.*

In the special case discussed by Bernheim (1984) where the demand and cost functions are linear, the competitive output is double the monopoly output (as is well-known). Hence in that case, if there are three or more firms in the industry, then the rationalizable set is $[0, x^M]$.

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