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BARGAINING WITHOUT CONVEXITY
Generalizing the Kalai–Smorodinsky Solution *

T.C.A. ANANT and Badal MUKHERJI
Dehli School of Economics, Dehli 110007, India

Kaushik BASU
Princeton University, Princeton, NJ 08544, USA

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It is shown that Kalai and Smorodinsky's result – that their solution is the only one satisfying certain reasonable axioms – is true even if we generalize the domain of bargaining games to allow for non-convex utility feasibility sets. The paper defines a property called 'Nash equilibrium (henceforth NE) regularity' and shows that the result is true as long as the feasibility sets happen to be NE-regular.

1. Introduction

Most two-person bargaining models proceed on the assumption that the feasibile set is convex. If convexity is not naturally satisfied, a reference is usually made to convexification via lotteries. This route is however not necessary. The purpose of this paper is to show that Kalai and Smorodinsky's (1975) [1] theorem can be generalized to accommodate a large class of non-convex feasible sets. This is not possible for the Nash bargaining proposition. Hence, this ability to grapple with a larger domain is an additional strength of the Kalai–Smorodinsky solution.

We shall follow Kalai and Smorodinsky and define a *bargaining game* as an ordered pair (a, S), where $a \in \mathbb{R}^2$ and $S \subset \mathbb{R}^2$. Let U be the collection of all bargaining games, (a, S), such that (i) there exists $x \in S$ such that [2] $x \gg a$ and for all $x \in S$, $x \geq a$, and (ii) S is compact and convex. Kalai and Smorodinsky define a solution as a function $\phi : U \to \mathbb{R}^2$ such that, for all $(a, S) \in U$, $\phi(a, S) \in S$. They then show that there is a unique solution (now known as the KS-solution) which satisfies four axioms.

Our aim in this paper is to widen the domain of the solutions and show that the KS-solution is still the unique one satisfying the same four axioms.

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[1] See also Luce and Raiffa (1957) and Friedman (1986).

[2] We use $x \gg y$ to mean $x_i > y_i$, $i = 1, 2$; $x \geq y$ to mean $x_i \geq y_i$, $i = 1, 2$; and $x > y$ to mean $x \geq y$ and there exists i such that $x_i > y_i$.

2. Convexity and NE-regularity

Let X be a compact subset of $\mathbb{R}^2$. Let

$$M_1(X) \equiv \max\{x \mid (x, y) \in X\} \quad \text{and} \quad M_2(X) \equiv \max\{y \mid (x, y) \in X\},$$

and

$$m_1(X) \equiv \max\{x \mid (x, M_2(X)) \in X\} \quad \text{and} \quad m_2(X) \equiv \max\{y \mid (M_1(X), y) \in X\}.$$

These are illustrated in fig. 1.

We shall say that f is an *NE-function* of X if $f: [m_1(X), M_1(X)] \to \mathbb{R}$ such that for all $x_1 \in [m_1(X), M_1(X)]$, $f(x_1) = \max\{x_2 \mid (x_1, x_2) \in X\}$. We shall use f_x to denote the NE-function of X.

A set $X \subset \mathbb{R}^2$ is defined as *NE-regular* if it is connected, compact and its NE-function is strictly decreasing and continuous. Let the shaded area, which is bounded on the north-east by the rectangular hyperbola, $x_1 x_2 = 4$, be denoted by S. S includes all the boundary points. It is easy to verify that S is NE-regular.

Figure 2 illustrates cases where the set are not NE-regular. The pictures, with the NE-functions thickened, are self-explanatory.

Lemma 1, below, shows that if a set is compact and convex it must be NE-regular. Clearly then Lemma 1 in conjunction with fig. 1 establish that the collection of all compact and convex sets in $\mathbb{R}^2$ is a proper subset of the collection of all NE-regular set in $\mathbb{R}^2$.

Lemma 1. If $X \subset \mathbb{R}^2$ is compact and convex, it must be NE-regular.

Proof. Suppose X is compact and convex but not NE-regular. Compactness ensures that f_x is well defined. It is easy to check that if f_x is discontinuous then X must be non-convex. Hence the only thing that remains to be proved is that f_x is strictly decreasing.

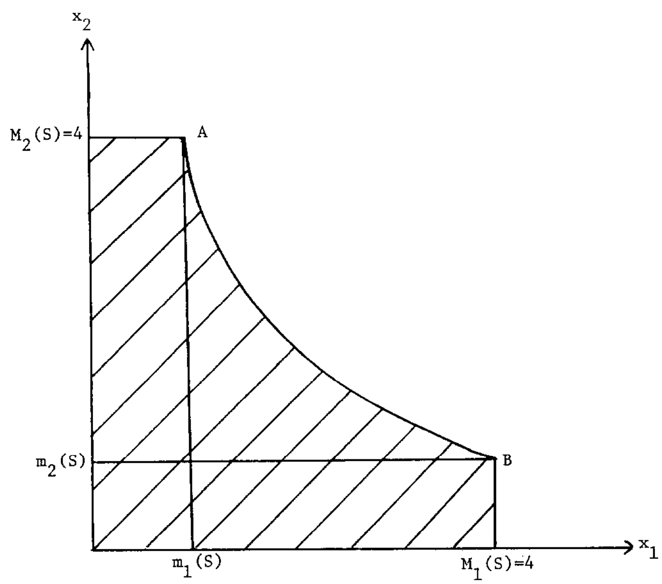

Fig. 1.

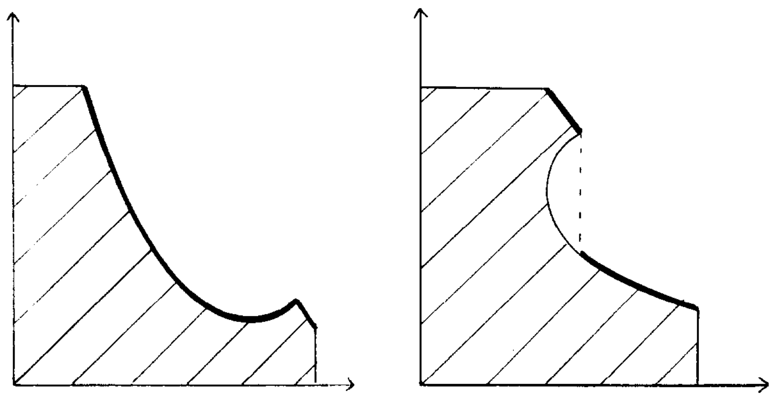

Fig. 2.

Let $m_1(X) < a < b < M_1(X)$. From the definition of $m_1(X)$, we know $f_x(m_1(x) > f_x(a)$. Let $\lambda \in (0, 1)$ be such that $\lambda m_1(X) + (1-\lambda)b = a$. From the convexity of X it follows that

$$(\lambda m_1(X) + (1-\lambda)b,\ \lambda f_x(m_1(X)) + (1-\lambda)f_x(b)) \in X.$$

From the definition of the NE-function, we have $f_x(a) \geq \lambda f_x(m_1(X)) + (1-\lambda)f_x(b)$. Since $f_x(m_1(X)) > f_x(a)$, it follows $f_x(a) > f_x(b)$. □

3. The bargaining problem

We shall proceed to directly widen the domain of the Kalai–Smorodinsky solution of the bargaining problem.

Let W be the collection of all bargaining games, (a, S), such that: (1) there exists $x \in S$ such that $x \gg a$ and for all $x \in S$, $x \geq a$, and (2) S is NE-regular. Comparing (1) and (2) with (i) and (ii) in section 1 and referring to fig. 1 and Lemma 1 it is immediate that W is a strictly larger collection than U. For example, a game with feasible set as in fig. 1 would belong to W but not to U.

A *general solution* is a function $\phi: W \to \mathbb{R}^2$ such that, for all $(a, S) \in W$, $\phi(a, S) \in S$.

The properties of general solutions which are proposed by Kalai and Smorodinsky (1975) are now stated as axioms without discussions. The reader who needs arm-twisting to be persuaded is referred to their paper. Actually, the first three axioms are the same as used by Nash and only the fourth axiom is specific to Kalai and Smorodinsky.

Axiom P (Pareto). For every $(a, S) \in W$, there is no $y \in S$ such that $y > \phi(a, S)$.

Axiom S (Symmetry). Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ such that $T((x_1, x_2)) = (x_2, x_1)$. For every $(a, S) \in W$, $\phi(T(a), T(S)) = T(\phi(a, S))$.

Axiom C (Cardinality). If $(a, S), (a', S') \in W$ and there exists real numbers d_i (> 0) and c_i, $i = 1$, 2, such that $a_i = c_i + d_i a_i'$ and $x \in S$ if and only if there exists $x' \in S'$ such that $x_i = c_i + d_i x_i'$, for $i = 1, 2$, then $\phi_i(a, S) = c_i + d_i \phi_i(a', S')$.

Axiom M (Monotonicity). If $(a, S) \in W$, define function g_s as follows:

$$g_s(x) = \begin{cases} f_s(x), & \text{if } x \in [m_1(S), M_1(S)], \\ M_2(S), & \text{otherwise.} \end{cases}$$

If $(a, S_1), (a, S_2) \in W$ such that $M_1(S_1) = M_1(S_2)$ and $g_{s_1} \leq g_{s_2}$, then $\phi_2(a, S_1) \leq \phi_2(a, S_2)$.

Definition 1. A general solution, ϕ, will be described as a KS-solution if for every $(a, S) \in W$, $\phi(a,S)$ is the vector-undominated element in the intersection of S with the line segment joining points a and $(M_1(S), M_2(S))$.

Our central theorem simply states that even in the enlarged domain W, Axioms P, S, C and M are equivalent to the KS-solution.

Remark 1. It is interesting to note that since NE-regularity implies the continuity of the associated NE-function, it follows that the cardinality axiom can be relaxed to first-difference comparability without hurting our main theorem [see Basu (1983)].

Remark 2. In proving Theorem 1 large parts follow a method similar to Kalai and Smorodinsky's proof. We skip over these parts with remarks like 'it is easy to see', and concentrate instead on what is novel.

Remark 3. A related paper by Conley and Wilkey (1989) [3] examines the KS-solution in games with free disposal (which is clearly a larger domain than the collection of NE-regular sets). They show that if the Pareto axiom is weakened to the Weak Pareto axiom then (given the other axioms) we get back the KS-solution. the use of Weak Pareto solution in a bargaining context may however be questioned on the ground that it requires player A to cooperate with B, who refuses to cooperate with A even when such cooperation can enhance A's welfare without lowering B's welfare. It is therefore interesting to see how much convexity can be weakened without losing the Strong Pareto axiom. Our concept of NE-regular bargaining sets and the theorem that follows answer precisely this question.

Theorem 1. A general solution satisfies Axioms P, S, C and M if and only if it is the KS-solution.

Proof. That the KS-solution exists can be proven as follows:

Denoting the NE-function of S by f_x and noting that

$$f_s(m_1(S)) = M_2(S) \quad \text{and} \quad f_s(M_1(S)) = m_2(S),$$

we have

$$f_s(m_1(S)) - a_2 \geq [m_1(S) - a_1][M_2(S) - a_2]/[M_1(S) - a_1]$$

and

$$f_s(M_1(S)) - a_2 \leq [M_1(S) - a_1][M_2(S) - a_2]/[M_1(S) - a_1].$$

[3] We are grateful to Prof. William Thompson for bringing this to our attention.

Using the continuity of f_s and applying a standard fixed point argument we get that there exists

$$x^* \in [m_1(S),\ M_1(S)] \quad \text{s.t.} \quad f_s(x^*) - a_2 = [x^* - a_1][M_2(S) - a_2]/[M_1(S) - a_1].$$

It is clear that the KS-solution picks the point $(x^*, f_s(x^*))$ in (a, S). Also it is easy to see that this solution satisfies Axioms S, C and M. As for Pareto optimality, it follows from the fact f_s is stictly downward sloping.

Now, let ϕ^* be a general solution which satisfies Axioms P, S, C and M. The proof is completed by showing that ϕ^* is the KS-solution. Given Axiom C, without loss of generality we may restrict our attention to *normalised bargaining games* in W, that is, $(a, s) \in W$ such that $a = 0$ and $M_1(S) = M_2(S) = 1$. Let $(0, S) \in W$ be a normalised bargaining game.

Define $\hat{S} \equiv \{x \in \mathbb{R}^2 \mid x \geq 0,\ x \leq y$, for some $y \in S]$. Clearly $\hat{S}$ is NE-regular. By applying Axiom M, we get $\phi^*(0, S) = \phi^*(0, \hat{S})$. Next define

$$Y = \{x \in \mathbb{R}^2 \mid (x_1, x_2) \in \hat{S} \text{ or } (x_2, x_1) \in \hat{S}\}.$$

If we daub ink all over $\hat{S}$ and fold the page along the 45° line through the origin and again reopen the page, then the area now covered in ink is given by Y. Note that Y is symmetric around the 45° line.

We shall prove that y is a NE-regular set. Let f_y be the NE-function of Y. It is easy to check that $f_y : [m_1(Y),\ M_1(Y)] \to \mathbb{R}$ such that for all $x \in [m_1(Y),\ M_1(Y)]$, $f_y(x = \max\ \{f_{\hat{s}}(x),\ f_{\hat{s}}^{-1}(x)\}$. The existence of an inverse function of $f_{\hat{s}}$ is ensured by the fact that it is strictly decreasing. Since $f_{\hat{s}}$ is continuous, $f_{\hat{s}}^{-1}$ is continuous. Hence f_y is continuous. It is also clear that both $f_{\hat{s}}$ and $f_{\hat{s}}^{-1}$ are strictly decreasing. Hence, f_y is strictly decreasing. This establishes the NE-regularity of Y.

Hence, $(0, Y) \in W$. By Axioms P and S it follows that $\phi^*(0, Y)$ is a point on the intersection of the 45° line with the set Y which is vector-undominated by all points in this intersection. Since Y is a superset of $\hat{S}$ and $M_1(Y) = M_1(\hat{S})$ and $M_2(Y) = M_2(\hat{S})$, it follows by applying Axiom M twice that $\phi^*(0, Y) \geq \phi^*(0, \hat{S})$. But from the definition of Y we know that the highest point on the 45° line in Y coincides with the highest point on the 45° line in $\hat{S}$. By Axiom P we get $\phi^*(0, Y) = \phi^*(0, \hat{S})$. Hence $\phi^*(0, S) = \phi^*(0, Y)$. This implies that $\phi^*(0, S)$ is the highest point on the intersection of the 45° line with S, which is precisely the point that would be selected by the KS-solution. □

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