Economics Letters 23 (1987) 147–151
North-Holland

MODELING FINITELY-REPEATED GAMES WITH UNCERTAIN TERMINATION

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Received 8 October 1986
Final version received 20 November 1986

A supergame is modeled in which two agents play a game repeatedly and while the finiteness of the repetitions is common knowledge, the number of repetitions is not. Such games can yield non-standard results. This is illustrated with the Prisoner's Dilemma where a cooperative outcome is shown to be possible.

1. Introduction

Most formal models of finite supergames assume that it is common knowledge among the players as to when the repetitions will terminate. This excludes a large class of real-life interactions which, while finite, is characterised by uncertainty concerning the date of termination. Suppose a landlord and his tenant meet each year to play one round of the Prisoner's Dilemma and they know that they will do so till one of them dies. It is reasonable to assume that it is common knowledge that they will play this game a finite number of times. However it would be quite unrealistic to assume that the date of termination is common knowledge.

The purpose of this paper is to suggest a way of modeling repeated games where the *finiteness* is common knowledge but the *number* of repetitions is unknown. This is then applied to the Prisoner's Dilemma and it is shown that cooperation may occur under certain parametric configurations.

It appears at first sight that supergames with uncertain termination will give us no different results from the standard finite supergames. One possible line of reasoning is as follows. In the above example, clearly we can assume that both players know that they will not live for 500 years. Hence when playing the 499th game (if they do play that game) both will play non-cooperatively. Hence they will do the same in the 498th game; and by repeated use of this argument, it follows that they will not cooperate ever during their lifetime, thereby producing the kind of behavior predicted by the standard finitely-repeated Prisoner's Dilemma.

The fallacy in the above argument is this. Even though right now they know that they will not live for 500 years, if they do live for 499 years, at that time it is not clear that they will be sure that that is their last year. Hence, to use the above kind of backward induction, we need to pick a year such that in all years before that both know (and it is common knowledge) that that is their last year. Clearly there may not exist such a year on which the backward-induction argument can be anchored.

* I have benefited from a seminar at the Indian Statistical Institute, Delhi, and from some comments of Debraj Ray.

2. A formalisation

Let $G=(S_1, S_2, U_1, U_2)$ be a two-person non-cooperative game in strategic form. S_i is the set of strategies open to player i and U_i: $S_1 \times S_2 \to R$ is i's payoff function. For every positive integer T, we use $G(T)$ to denote a T-period repetition of the game G. $G(T)$ is a finite-horizon supergame. A strategy of player i in $G(T)$ is a specification of what move he will make after every possible history of play. [1] Let $S^i(T)$ be the set of strategies open to i in $G(T)$. Given strategies $s \in S^1(T)$ and $r \in S^2(T)$ we can work out in the usual way the outcomes in each of the T games. Then i's payoff is simply the summation of the payoff received in each game. Given a positive integer, T, if $s \in S^1(T)$ and $r \in S^2(T)$, we use $H^i(s, r)$ to denote i's payoff in the supergame $G(T)$, when s and r are the strategies used.

What we are interested in studying here is a special kind of finite supergame – one in which the players do not know when the repetitions will end. It will be assumed throughout that the repetitions actually end at stage N (i.e., G is played N times). Let $n(i)$ be the stage at which i expects the supergame to terminate. We assume pointwise expectations for simplicity.

For analysing the supergame $G(N)$ it is not enough to know $n(1)$ and $n(2)$ because i's behavior depends on what i believes about j's belief concerning when the repetitions will end. Definition 1 formalises the concept of a *system* of beliefs. But first we need some notation. For $i \in \{1, 2\}$, $j \in \{1, 2\} \setminus \{i\}$ and any positive integer t, $g(i, t)$ denotes the sequence $(i, j, i, j, \ldots)$ with t elements. For brevity we drop the commas when writing these sequences. Thus $g(2, 5) = (21212)$. Using I to denote the set of positive integers, define Z as follows:

$$Z = \{ g(i, t) \mid i \in \{1, 2\}, t \in I \}.$$

Definition 1. A mapping n: $Z \to I$ with $n(1)$, $n(2) \geqslant N$ will be referred to as a *termination conjectures mapping*.

The interpretation of this is as follows. If $n(1212) = t$, it means that 1 believes that 2 believes that 1 believes that 2 believes that the repetitions will end at stage t. Changes in conjectures during the game are not allowed. As will be clear from the remarks below, this is a harmless restriction in the present context. In a fuller treatment, this will of course have to be relaxed.

We assume $n(i) \geqslant N$ in order to avoid the question of what i believes and does when asked to play game $n(i)+1$. One way of handling this would be to assume that if i finds that the game is continuing after stage $n(i)$, he forms a fresh conjecture as to how long the game will continue. This raises some troublesome philosophical questions and my assumption is designed to skirt these. In my formulation, for $t \geqslant 2$, $n(g(i, t))$ may be less than N. It will be clear as we go along how such conjectures are handled.

If two players play G repeatedly for N periods and their beliefs about the termination of repetitions are summarised by the termination conjectures mapping n, we shall say that these agents are playing the *finite open-ended* (or just *open*) *supergame* $G(N, n)$. It is worth noting that if $n(x) = N$ for all x in Z, i.e., it is common knowledge that the game will terminate at N, then $G(N, n)$ is the usual N-period finitely-repeated game. Thus conventional finitely-repeated games are a special case of the class of games being considered here.

Before formally describing what outcomes we may expect from $G(N, n)$, we need to develop some notations and definitions. If $s \in S^i(R)$ and $T \leqslant R$, we use s/T to denote the restriction of s to T periods. Thus $s/T \in S^i(T)$.

[1] For more formal definitions see, for example, Benoit and Krishna (1985) or Friedman (1985).

Definition 2. $s^i \in S^i(T)$ is a *best response* to $s^j \in S^j(R)$ iff the following is true: [2]

(i) In case $T \leqslant R$, then $H^i(s^i, s^j/T) \geqslant H^i(r, s^j/T)$, for all $r \in S^i(T)$.
(ii) In case $T > R$, then there does not exist $r \in S^i(T)$, such that for all $\hat{s}^j \in S^j(T)$, such that $\hat{s}^j/R = s^j$, $H^i(r, \hat{s}^j) \geqslant H^i(s^i, \hat{s}^j)$ and for some $\hat{s}^j \in S^j(T)$, such that $\hat{s}^j/R = s^j$, $H^i(r, \hat{s}^j) > H^i(s^i, \hat{s}^j)$.

The interpretation of this definition is this. If i believes the game will end at T and i believes j believes the game will end at R and i believes j is fielding strategy s^j, then i will play his *best response* to s^j. If $T \leqslant R$ [i.e., case (i)], best response is defined in the usual way. If $T > R$, there is a problem. Since i believes that j believes that the game will end at stage R, it is not clear what kind of play i can anticipate from j in the games after stage R. Implicit in Definition 2 is the supposition that i believes that j may do anything in these remaining games.

In what follows, define k as a mapping on the set of positive integers with the following property: $k(t)$ is i, if t is odd; otherwise $k(t) = j$.

Definition 3. Given an open supergame $G(N, n)$, $s^i \in S^i(n(i))$ is an *equilibrium strategy* for player i iff there exists an infinite sequence $(r_2, r_3, \ldots)$ such that $r_2 \in S^j(n(ij))$ and s^i is a best response to r_2; and for all $t = 3, 4, \ldots$, $r_t \in S^{k(t)}(n(g(i, t)))$ and r_{t-1} is a best response to r_t. The sequence $(r_2, r_3, \ldots)$ will be referred to as a *rationalising* sequence for s^i. If s^1 and s^2 are equilibrium strategies for players 1 and 2, then $(s^1/N, s^2/N)$ is a *solution* of $G(N, n)$. The outcomes in the N games that occur from the agents playing the solution will be referred to as an *equilibrium outcome* of $G(N, n)$.

Remark 1. The above definition is an adaption of the idea of 'rationalizability' [Bernheim (1984), Pearce (1984)]. In dealing with extensive games it is possible to use a more refined solution concept which requires the best response property to be satisfied at every subgame. In an extensive form game it is reasonable to allow each player to revise his conjecture about (i) what strategy the other player is playing, and (ii) the other player's belief (and higher-order beliefs) about when the game will end. For simplicity we do not allow for such revisions in this paper. Fortunately, this does not matter too much because it is easy to state some sufficient conditions under which a player will not need to revise his beliefs (and these conditions are satisfied in what remains of this paper). Suppose s^1 and s^2 are equilibrium strategies and $(r_2, r_3, \ldots)$ and $(t_2, t_3, \ldots)$ are their rationalizing sequences, respectively. If $s^2/N = r_2/N$ and $s^1/N = t_2/N$ then if the players play s^1 and s^2, it is reasonable to assume that no player will need to revise his conjecture during the N plays.

3. The Prisoner's Dilemma [3]

Let $\hat{G}$ be the Prisoner's Dilemma game [see Friedman (1985)] with the additional assumption that a player gets a larger payoff if they both cooperate in two successive games than what he gets by defecting in both games while his opponent defects once and cooperates once. Since in $\hat{G}$, $S^1 = S^2 = \{\text{cooperate, defect}\}$, we drop the superscripts and let S denote each player's strategy set. Similarly, $S^i(T)$ is now written as $S(T)$.

[2] Here and elsewhere, when i and j occur in the same definition, $i \neq j$.
[3] Much of the analysis of this section can be carried over to the 'chain-store paradox' studied by Selten (1978).

If $n(x) = N$, for all x, then we have the standard finitely-repeated Prisoner's Dilemma for which the backward-induction works and the only equilibrium is defection in each game. [4] It is easy to see that the backward-induction result can be derived as a corollary of a more general theorem. [5]

Theorem 1. *If n is such that $n(g(i, t)) = M$, for all $i = 1, 2$ and for all $t = 1, \ldots, M$, where $M \geqslant N$, then the only solution of $\hat{G}(N, n)$ is one where both players defect throughout.*

What is interesting about an open supergame is that it is possible that both agents will play the Prisoner's Dilemma cooperatively. Note that the supergame is not just finite, but its finiteness is common knowledge.

Theorem 2. *If n is such that for all $i \in \{1, 2\}$ $N < n(i) < n(ij) < n(iji) < \ldots$, then cooperation in every game is an equilibrium outcome of $\hat{G}(N, n)$.*

Remark 2. If the hypothesis of Theorem 2 is valid and both players cooperate throughout, not only is that an equilibrium outcome but (and this follows from Remark 1) no player has any reason to revise his conjectures during the course of play.

Appendix

Proof of Theorem 1. It is easy to check that defecting everywhere is an equilibrium. What we show is that this is the only equilibrium. Let s^i be an equilibrium strategy for person i and let $(r_2, r_3, \ldots)$ be a rationalizing sequence. It is shown that s^i entails defecting in game $M - t$ (i.e., for all histories of play) for all t, $0 \leqslant t \leqslant M - 1$.

Suppose,

$$0 \leqslant t \leqslant M - 1. \tag{1}$$

From the hypothesis of the theorem we know that

$$S(n(i)) = S(n(ij)) = \ldots = S(n(g(i, M))) = S(M). \tag{2}$$

It is first shown that r_{t+1} entails playing defection in game M. (1) and (2) implies $r_{t+1} \in S(M)$. Hence for r_{t+1} to be a best response to r_{t+2} (or any strategy for that matter), r_{t+1} must entail defection in game M.

Next it is shown that r_t entails defecting in M and $M - 1$. By a similar logic as above, r_t entails defection in M. (1) and (2) implies $r_t \in S(M)$. Since r_t is a best response to r_{t+1} and r_{t+1} entails defection in game M, r_t must entail defection in game $M - 1$.

By a repeated use of the above argument we get

$$r_2 \text{ entails defection in games } M,\ M - 1, \ldots, M - (t - 1). \tag{3}$$

Since $s^i \in S(M)$ and s^i is a best response to r_2, (3) implies that s^i entails defection in game $M - t$. Q.E.D.

[4] For some attempts to explain cooperation (different from the one undertaken here) see Kreps, Milgrom, Roberts and Wilson (1982), Basu (1985), Binmore (1985) and Reny (1985).

[5] See Schick (1983) for some similar arguments and some philosophical objections to them.

Proof of Theorem 2. If $s^i \in S(R)$ and s^i is the strategy of playing tit-for-tat (TFT) till game $R-1$ and then defecting in R, then we shall refer to s^i as the TFT-R strategy. Let $s^i \in S(n(i))$ be a TFT-$n(i)$ strategy. Let $(s_2, s_3, \dots)$ be a sequence with the property that for all $t \geqslant 2$, $s_t \in S(n(g(i, t)))$ and s_t is a TFT-$n(g(i, t))$ strategy. It is easy to check that s^i is a best response to s_2, s_2 is a best response to s_3 and so forth. Hence, TFT-$n(i)$ is an equilibrium strategy. Since $n(1)$, $n(2) > N$, both players cooperating in each of the N games is an equilibrium outcome. Q.E.D.

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