Mathematical Social Sciences 14 (1987) 275-288
North-Holland

AXIOMS FOR A FUZZY MEASURE OF INEQUALITY

Kaushik BASU

The Delhi School of Economics, Delhi 110 007, India, and The Institute for Advanced Study, Princeton, NJ 08540, U.S.A.

Communicated by B. Dutta
Received 1 March 1986
Revised 2 October 1986

It has been argued that the concept of inequality is *inherently* imprecise. A difficulty with standard inequality measures is that they generally make no allowance for this, and when they do, it is by dropping the 'completeness' axiom in ranking social states (e.g. the Lorenz criterion). It is argued here that the erring axiom is not 'completeness' but 'exactness' which, being implicit, tends to escape notice. A fuzzy measure of inequality, along with a set of necessary and sufficient axioms, is established. The new measure has several attractive properties: It allows for tentative judgements and doubts. It is easy to interpret and compute, and the Gini ranking turns out to be a nearest exact approximation of it.

Key words: Inequality; fuzzy ranking; Gini coefficient.

1. Motivation

Human perceptions in making value judgements are often nebulous. Thus in comparing the extent of inequality in two social states we frequently find ourselves unable to reach a clear answer. The conventional attitude to this – for instance, in traditional cost-benefit analysis – is to ignore such doubts, commit ourselves to some rule or measure and to somehow rank the two states. This is an unattractive method. As Sen (1973) argues persuasively, the notion of inequality may be inherently imprecise and to do away with the imprecision by brute force may not be the best approach. After all, a measure of inequality is meant to capture our inherent attitude to inequality and if the latter is imprecise, then this ought to be reflected in the measure. This led Sen and others to argue the case for using quasi-orderings over social states. A quasi-order is a reflexive and transitive binary relation. It need not be complete or connected, that is, it is free to withhold judgement over some pairs of social states.

The use of quasi-orders in welfare economics in general has a long historical tradition dating back at least to Pareto, though its use in inequality ranking and cost-benefit analysis has become popular only in recent times (see, e.g., Sen, 1973; Dasgupta, 1982). However, this method has a tendency to err on the side of caution. Doubts are converted into complete reticence. Also, it has an artificial sharpness –

0165-4896/87/$3.50

over every pair of states it either gives a clear verdict or says nothing. There is no room for judgements of different degrees of tentativeness.

Recent advances in the theory of fuzzy sets enable us to talk of human imprecisions in a meaningful way. And it is not difficult to maintain that a fuzzy binary relation captures our ambivalence in ranking states according to inequality better than a quasi-ordering. Hence, the present paper is a search for a suitable fuzzy measure of inequality.[1] Using the framework of Gini and Lorenz, one such measure is axiomatically derived. This is done in Section 4 which is the focal point of this paper. The sections following 4 examine the properties of the new measure. One interesting property is that the conventional Gini-ranking is, in a sense, a best unfuzzy approximation of our fuzzy measure. Section 2 briefly recapitulates the Gini and Lorenz relations and highlights, in particular, those shortcomings which a fuzzy relation can overcome. Section 3 lays out the concepts of fuzzy set theory which are used in this paper.

2. The Gini and Lorenz rankings

In order to focus exclusively on the distribution problem we consider a society of n individuals with aggregate income fixed at T units. Income cannot be negative and given any array of incomes we always rearrange elements so that they increase monotonically. Thus the set, X, of alternatives is defined as:

$$X=\left\{x\in R^n \,\middle|\, 0\le x_1\le x_2\le\cdots\le x_n,\ \sum_{i=1}^{n} x_i=T\right\},$$

where R^n is the n-dimensional Euclidean space. $x\in X$ will be referred to as a social state, an income distribution or an income vector; and x_i is the income of the ith poorest person (ties being broken arbitrarily) in social state x. The social state in which everybody receives the same income is referred to as the *egalitarian state*. The symbol e is reserved for such a state. Thus $e_i=T/n$, for all i. We denote the Cartesian product of X without the diagonal elements by Ω:

$$\Omega=X\times X\setminus\{(x,x)\mid x\in X\}.$$

An important purpose of an inequality measure is to generate a ranking over X. We begin by considering the Gini and Lorenz rankings. These are best explained in terms of the Lorenz curve. Briefly, the Lorenz curve of a social state shows, for each

[1] An 'inequality measure' is an index of the extent of inequality in each social state. Hence, given an inequality measure, we can rank social states on the basis of the amount of inequality in each of them. In this paper the expression 'inequality measure' is (almost always) used to refer directly to such a ranking. The term 'fuzzy inequality measure' is used accordingly.

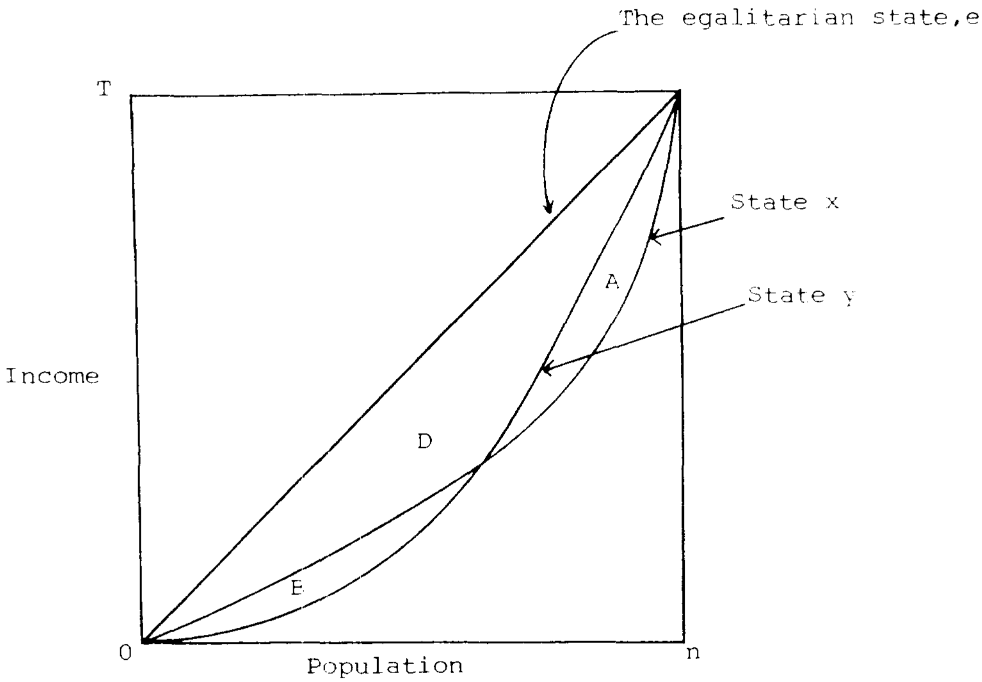

Fig. 1.

number of the poorest people in the population, the amount of income received by them.[2] Fig. 1 shows the Lorenz curves of states x and y.

According to the *Lorenz ranking*, state z is more equitable than w if the Lorenz curve of z dominates (i.e. is somewhere above and nowhere below) the Lorenz curve of w. The *Gini coefficient* of inequality of z is equal to the area between the Lorenz curves of e and z divided by $nT/2$. State z is more equitable than w according to the *Gini ranking* if the Gini coefficient of z is smaller than that of w.

Let G and L denote the binary relations representing greater equality in terms of, respectively, the Gini and Lorenz criteria. We use a slightly unconventional notation for binary relations: R is a binary relation on Ω means $R: \Omega \to \{0, 1\}$. Thus instead of writing xRy or *not* xRy, as is conventional, we write, respectively, $R(x, y) = 1$ or $R(x, y) = 0$. Also note that R is defined on Ω and not on the more common $X \times X$. This does no harm. Since we are confining our attention to inequality relations, it is clear that if R is the weak relation, xRx would always be true; and if it is the strict relation, xRx would never be true. Non-distinct pairs are clearly uninteresting and we may safely omit them from our domain of discourse, and concentrate on Ω – the set of all distinct pairs of states.

[2] Strictly speaking the axes of the Lorenz diagram should show the *proportion* of population and *proportion* of aggregate income. Our method of showing population and income, instead, does no harm since n and T are fixed throughout the paper.

In the context of Fig. 1, clearly

$$G(x,y)=1 \leftrightarrow D+A<D+B$$
$$\leftrightarrow \quad A<B; \qquad \text{and}$$
$$L(x,y)=1 \leftrightarrow A=0 \text{ and } B>0.$$

G and L may now be formally defined. For all $x\in X$, let $\hat{x}_j \equiv x_1+\cdots+x_j$, for all $j\leq n$. For all $x,y\in X$, define

$$E_{xy}=\sum_{j=1}^{n}\max\{\hat{x}_j-\hat{y}_j,0\}.$$

E_{xy} is the area of x's dominance over y in the Lorenz diagram.[3] In Fig. 1, $E_{xy}=B$ and $E_{yx}=A$.[4]

Now for the formal definition: For all $(x,y)\in\Omega$,

$$G(x,y)=\begin{cases}1, & \text{if } E_{xy}>E_{yx}\\ 0, & \text{otherwise}\end{cases}$$

$$L(x,y)=\begin{cases}1, & \text{if } E_{xy}>E_{yx}=0\\ 0, & \text{otherwise.}\end{cases}$$

Let $\hat{G}$ denote the following relation: For all $(x,y)\in\Omega$,

$$\hat{G}(x,y)=1\leftrightarrow E_{xy}=E_{yx}.$$

Let $\tilde{G}=G+\hat{G}$. $\tilde{G}$ will be referred to as the *weak Gini(E)-relation* and G as the *strong Gini(E)-relation*. A binary relation R on Ω is *complete* or *connected* if and only if for all $(x,y)\in\Omega$, either $R(x,y)=1$ or $R(y,x)=1$. Note that $\tilde{G}$ is complete.

Both the Lorenz and Gini criteria have received extensive attention in the literature and they have been interpreted, axiomatised and modified many times over.[5] The critique of $\tilde{G}$ which is of relevance to us here is its completeness property. As already argued above with reference to Sen (1973, Ch. 3), the completeness of an inequality ranking does serious injustice to our inherent idea of inequality.

In response, economists have often turned towards incomplete relations. One such relation is L. Whenever the Lorenz curves of x and y intersect, we have

[3] Taking a cue from the theorem of Dasgupta, Sen and Starrett (1973), it is possible to think of E_{xy} as the minimum volume of equalizing transfers effected on y so as to create a distribution which Lorenz dominates x.

[4] This is approximately so in the case where population is finite and the two Lorenz curves intersect at noninteger values of the population. As the population becomes larger the geometric representation (i.e. A and B) converges to the formal definition (i.e. E_{yx} and E_{xy}). I owe this observation to a referee. This 'approximate' Gini relation, based on E_{xy} and E_{yx}, will be referred to in this paper as the Gini(E) relation.

[5] For instance, Ranadive (1965), Atkinson (1970), Sen (1974), Pyatt (1976), Takayama (1979), Weymark (1981), Thon (1982), Tendulkar (1983).

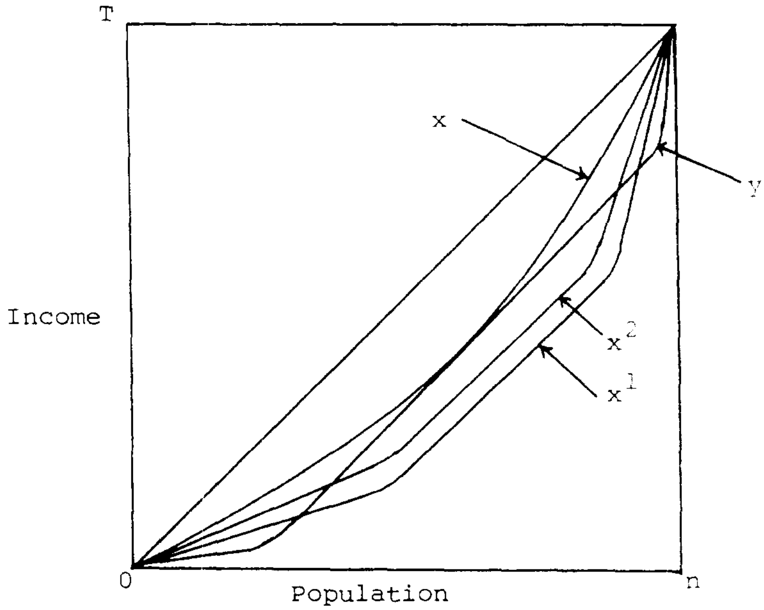

Fig. 2.

$L(x, y) = L(y, x) = 0$. As pointed out above, a measure such as this errs on the side of caution. There are two other related shortcomings to which attention ought to be drawn. First, L fails to distinguish between major and minor conflicts. For example, in Fig. 2, though social state x^2 has a much better distribution than state x^1 (this is so in terms of the Lorenz criterion itself), the Lorenz criterion ranks x^1 and y the same way as x^2 and y. Namely, $L(x^i, y) = L(y, x^i) = 0$, for $i = 1$ and 2.

Secondly, L entails a perceptional discontinuity. Consider a sequence, $\{x^n\}$ of social states which converge to state x from below. Let x Lorenz-dominate y. The first 2 elements of the sequence $\{x^n\}$, as well as x and y are shown in Fig. 2. As will be obvious from the figure, it is quite possible that $L(x^n, y) + L(y, x^n) = 0$, for all n; and $L(\lim x^n, y) + L(y, \lim x^n) = L(x, y) + L(y, x) = 1 \neq 0$. There is, in other words, a sudden jump from 'no judgement' to 'full judgement'.[6]

It is clear that while the critiques of traditional orderings of social states, like the Gini criterion, are extremely persuasive, the answers that have been given in response are unsatisfactory. It is the contention of this paper that while the standard response to the critique of using orderings has been to jettison the 'completeness' axiom, the real culprit is the 'exactness' axiom which is generally implicit and therefore has managed to slip by unnoticed. The exactness axiom becomes obvious once we approach binary relations from the larger framework of *fuzzy* binary relations.

[6] Other kinds of incomplete rankings have been considered in the inequality literature though much of these criticisms would be applicable even there (see Sen, 1973, Ch. 3 and Basu, 1984).

3. Fuzzy sets and relations

Let U be the universal (unfuzzy) set. Then B is a fuzzy subset or simply a *fuzzy set* implies $B: U \to [0,1]$. $B(x)$ represents the extent to which x belongs to B. If $B(U) \subset \{0,1\}$ then B is an *exact set* or simply a set. If B is exact we may write $x \in B$ and $x \notin B$ instead of $B(x)=1$ and $B(x)=0$.

If X is an exact set of alternatives then a *fuzzy binary relation* (FBR) on $W \subset X \times X$ is a mapping $R: W \to [0,1]$. An FBR, R, on W is *reflexive* iff for all $(x,x) \in W$, $R(x,x)=1$. It is *complete* iff $(x,y),(y,x) \in W$ implies $R(x,y)+R(y,x) \geq 1$. It is *E-transitive*[7] iff $(x,y),(y,z),(x,z) \in W$ and $R(x,y)=R(y,z)=1$ implies $R(x,z)=1$. If R is reflexive, complete, E-transitive and exact then R is an *ordering* or an exact ordering. This makes it clear that if we find an ordering too demanding, it is not necessary to drop completeness. We could instead drop the exactness requirement. Since I shall work with the strong preference relation most of the time, an important property is 'asymmetry'. The FBR, R, on W is *asymmetric* iff for all $(x,y),(y,x) \in W$, $R(x,y)+R(y,x) \leq 1$.[8] This corresponds to what in a probabilistic framework has been referred to as 'strict probabilistic judgement' (Barbera and Valenciano, 1983). If R is exact, our definition of asymmetry coincides with the conventional definition.

4. Axioms and the theorem

Both the Gini and Lorenz ranking over any pair $\{x,y\}$ depends only on E_{xy} and E_{yx}. Such an approach, i.e., one where the inequality relation over any pair $\{x,y\}$ depends only on E_{xy} and E_{yx} will therefore be referred to as the *Gini–Lorenz approach*. In searching for a suitable fuzzy relation we shall, in this paper, restrict attention to the Gini-Lorenz framework. In other words, if P is a fuzzy relation on Ω representing 'is-more-equitable-than', then there exist a mapping $\phi: R_+^2 \to [0,1]$, such that for all $(x,y) \in \Omega$,

$$P(x,y) = \phi(E_{xy}, E_{yx}). \tag{2}$$

[7] While the definitions of reflexivity and completeness in a fuzzy framework carries over quite obviously from the traditional exact approach, transitivity is difficult to interpret. Not surprisingly, transitivity has been defined in various ways (see, Orlovsky, 1975; Dubois and Prade, 1980; Basu, 1984). Dutta, Panda and Pattanaik (1986) have examined the relation between some of these alternative definitions and have, in particular, established an extremely useful theorem. In order to stay away from these controversies, I use here a very weak transitivity property. E-transitivity asserts that traditional transitivity is valid over domains where preferences are exact.

[8] There can be other ways of capturing the idea of asymmetry in a fuzzy context. Dutta (1987) requires his strict preference relation to satisfy the following 'antisymmetry' property: $P(x,y)>0$ implies $P(y,x)=0$. In the context of inequality ranking this does not seem appropriate. According to this, the smallest suspicion that x is more unequal than y disallows one from entertaining *any* possibility of y being more unequal than x. My definition makes the weaker assertion that great confidence that x is more unequal than y disallows one from having great confidence that it is the other way around.

We shall refer to such a binary relation P as a fuzzy inequality relation. R_+^2 is used to denote the non-negative orthant of the 2-dimensional Euclidean space.[9] By specifying ϕ differently we can get different fuzzy inequality relations. Clearly, G and L – as defined in (1) – are special cases of (2).

Instead of directly specifying (2), we shall proceed by imposing axioms on it. P being a strong relation, it would be asymmetric. We shall, for simplicity, rule out indifference over *distinct* pairs (x, y) and assume P to be complete on Ω. All these observations are summed up in our first axiom.

Axiom 1. *For all* $(x, y) \in \Omega$, $P(x, y) + P(y, x) = 1$.

It seems reasonable that if one state is completely egalitarian and the other not, we would unambiguously declare the former to be more equitable than the latter.

Axiom 2. *If* $y \in X$ *and* $y \neq e$, *then* $P(e, y) = 1$.

Finally, we assume P to be 'weakly cardinal', that is, it is required to preserve order over first differences in the terms E_{xy}. Let us state the axiom formally before discussing it. Before doing so it is useful to have a definition. Define

$$T(S) \equiv \{r \equiv (r_1, r_2) \in R_+^2 \mid r_1 + r_2 = S\}.$$

Clearly $T(S)$ is a simplex in R^2.

Axiom 3. *If* $r, v, t, u \in T(S)$ *for some* S, *then*

$$r - v \geq t - u \rightarrow \phi(r) - \phi(v) \geq \phi(t) - \phi(u).$$

To appreciate Axiom 3 intuitively, first consider the ordinal property:

(i) If $E_{xy} + E_{yx} = E_{zw} + E_{wz}$ then

$$E_{xy} \geq E_{zw} \rightarrow P(x, y) \geq P(z, w).$$

Since E_{xy} is the area of dominance of x over y (see Fig. 1), $E_{xy} + E_{yx}$ is the aggregate area of dominance between x and y. (i) asserts that if the aggregate area of dominance between (x, y) and that between (z, w) are the same and the area of x's dominance over y is greater than z's dominance over w, then $P(x, y) > P(z, w)$. This is clearly reasonable. Briefly, (i) asserts that $P(\cdot)$ preserves order over the E_{xy}'s.

[9] If we wish, we may restrict the domain of ϕ as follows: Let H be the largest value that $E_{xy} + E_{yx}$ can take as we vary x and y in X. Define $B \equiv \{r \in R_+^2 \mid r_1 + r_2 \leq H\}$. If ϕ is defined on B instead of R_+^2, our theorem is unaffected and, in fact, nothing of consequence changes. Note however that the collection of values that (E_{xy}, E_{yx}) can take as we vary x and y is a proper subset of B. Hence whether we define ϕ on R_+^2 or B, we are using some kind of a 'unrestricted domain' assumption. I have in a different context (Basu, 1983) argued at length the advantages of leaving the domain unrestricted in this manner.

Let us strengthen this to require the preservation of order not only over the E_{xy}'s but also the first differences of E_{xy}'s. That is,

(ii) If $E_{xy}+E_{yx}=E_{zw}+E_{wz}=E_{ab}+E_{ba}=E_{cd}+E_{dc}$ then $E_{xy}-E_{zw}\geq E_{ab}-E_{cd}\rightarrow P(x,y)-P(z,w)\geq P(a,b)-P(c,d)$.

It is easy to check that (ii) implies (i).[10] (ii) is a cardinality property. Axiom 3 asserts (ii) but requires it to be satisfied over the entire domain of the mapping ϕ. The properties and role of such a cardinality axiom in utility theory were widely debated in the thirties beginning with Lange's (1933) paper and I have discussed them formally in Basu (1982).

What is interesting is that there is only one fuzzy inequality relation which satisfies the above three axioms. This is stated in the theorem below.

One must be careful in interpreting the theorem. The fuzzy inequality relation being proposed in this paper is intuitive and seems to be of some intrinsic interest. Hence the axioms are not meant to be a justification of the fuzzy measure but a characterization of it.

The proof of the theorem is vastly simplified by making use of the lemma on affine functions stated below. This lemma is an immediate consequence of Theorem 1 in Basu (1983).

Lemma 1. *Let f be a real-valued mapping on an interval $[a,b]$ which satisfies the following property: For all t_1,t_2,t_3,t_4 in $[a,b]$, $t_1-t_2\geq t_3-t_4\rightarrow f(t_1)-f(t_2)\geq f(t_3)-f(t_4)$. Then there exist real numbers α and β, with $\beta>0$, such that for all t in $[a,b]$, $f(t)=\alpha+\beta t$.*

Now we are in a position to state and prove the main theorem.

Theorem. *The only fuzzy inequality relation satisfying Axioms 1, 2, and 3 is the following: For all $(x,y)\in\Omega$, $P(x,y)=E_{xy}/(E_{xy}+E_{yx})$.*

Proof. Consider the mapping ϕ in (2).

If $r\in T(S)$, then

$$\begin{aligned}\phi(r)&=\phi(r_1,r_2),\\&=\phi(r_1,S-r_1),\\&\equiv\psi_S(r_1).\end{aligned}\tag{3}$$

Thus ψ_S is a mapping from $[0,S]$ to $[0,1]$ and it is defined as in (3).

[10] Suppose in (ii), $E_{ab}=E_{cd}$. Then since $E_{ab}+E_{ba}=E_{cd}+E_{dc}$, (2) implies that $P(a,b)=P(c,d)$. Hence (ii) reduces to the condition:

$$E_{xy}-E_{zw}\geq 0\rightarrow P(x,y)-P(z,w)\geq 0,$$

which is the same as (i). Thus (ii) $\rightarrow$ (i).

Let us assume that Axioms 1–3 are valid. Given Axiom 3 and a real number S, the hypothesis of Lemma 1 is valid for ψ_S. Hence Lemma 1 implies that for each S, there exists $\alpha(S)$ and $\beta(S)$, with $\beta(S)>0$, such that for all t in $[0,S]$, $\psi_S(t)=\alpha(S)+\beta(S)t$. Hence,

$$\forall(x,y)\in\Omega,$$

$$P(x,y)=\beta(E_{xy}+E_{yx})E_{xy}+\alpha(E_{xy}+E_{yx}). \tag{4}$$

The next step is to prove that $\forall(x,y)\in\Omega$, $\alpha(E_{xy}+E_{yx})=0$. Note first that $\forall(x,y)\in\Omega$, $\exists w\in X$ such that $E_{ew}=E_{xy}+E_{yx}$. This may be seen as follows:

$$\begin{aligned}E_{xy}+E_{yx}&=\sum_{j=1}^{n}|\hat{x}_j-\hat{y}_j|\\&=\sum_{j=1}^{n-1}|\hat{x}_j-\hat{y}_j|,\quad\text{since }\hat{x}_n=\hat{y}_n=T.\\&\le\sum_{j=1}^{n-1}\left|j\frac{T}{n}-0\right|,\quad\text{since }\forall j,\ \hat{x}_j\le j\frac{T}{n}\text{ and }\hat{y}_j\ge 0.\end{aligned}$$

Therefore we can always find a positive number $\theta\le T/n$ such that

$$E_{xy}+E_{yx}=\sum_{j=1}^{n-1}\left|j\frac{T}{n}-j\theta\right|=E_{ew},$$

where $w=[\theta,\dots,\theta,T-(n-1)\theta]$.
Now,

$$\begin{aligned}\text{Axiom }2&\rightarrow P(e,w)=1\\&\rightarrow P(w,e)=0\text{ by Axiom 1.}\end{aligned}$$

(4) implies $P(w,e)=\beta(E_{ew})E_{we}+\alpha(E_{ew})$. Since $E_{we}=0$ and $P(w,e)=0$, $\alpha(E_{ew})$ must equal 0. Hence, $\alpha(E_{xy}+E_{yx})=0$. Therefore, (4) reduces to the following:

$$\forall(x,y)\in\Omega,\ P(x,y)=\beta(E_{xy}+E_{yx})E_{xy}. \tag{5}$$

By applying Axiom 1, it is easy to see that

$$\forall(x,y)\in\Omega,\quad \beta(E_{xy}+E_{yx})=1/(E_{xy}+E_{yx}).$$

The proof is completed by substituting this in (5). □

In the remainder of this Section I comment on some features of the new inequality ranking established in the above theorem. Let us begin by writing it out in full.

$$\forall(x,y)\in\Omega,\ P(x,y)=\sum_{j=1}^{n}\max\{\hat{x}_j-\hat{y}_j,0\}\bigg/\sum_{j=1}^{n}|\hat{x}_j-\hat{y}_j|. \tag{6}$$

From here on a reference to the 'fuzzy inequality relation' or the 'fuzzy measure' should be taken to mean (6).

A diagrammatic interpretation of the fuzzy inequality relation is easy and useful. Essentially (6) asserts that the extent to which x is more equitable than y is given by the area of x's dominance over y in the Lorenz curve diagram, divided by the aggregate of x's dominance over y and y's dominance over x. In terms of Fig. 1,

$$P(x,y) = B/(A+B).$$

Note that if x Lorenz-dominates y then $P(x,y)=1$. In fact, the fuzzy inequality relation satisfies the following property:

$$\forall (x,y) \in \Omega,\ L(x,y) = 1 \leftrightarrow P(x,y) = 1. \tag{7}$$

Hence our fuzzy inequality relation subsumes the Lorenz relation.

P being a strong preference relation, we would expect it to satisfy asymmetry and transitivity. It satisfies asymmetry of course because this was built into Axiom 1. As far as 'transitivity' goes, as already discussed (see footnote 7), it is far from clear as to what an appropriate definition ought to be in the context of *fuzzy* relations. Let us here consider E-transitivity (defined in Section 3) since this is an essential element in any formulation of transitivity. That the fuzzy inequality relation satisfies E-transitivity is obvious from (7) and the fact that L is a transitive exact relation.

In this paper we have ruled out the possibility that two distinct social states can be equally unequal. I offer two comments in defence. First, while it is true that this assumption, made in the context of exact measures, would rule out all continuous inequality measures, no such discontinuity problem arises here because of our assumption of fuzziness. Secondly, there is an intuitive justification for our assumption. Note first that the way x has been defined, if one society's income distribution is a permutation of that of the other, then these two states will not be considered distinct. Hence, if x and y are distinct states, it means that there will be some 'regions' (e.g. among the poorer people) where x has more inequality than y or *vice versa*. In such an event, to assert that x and y have the same amount of inequality implies that the extent to which x is more unequal than y where x is more unequal than y, exactly matches the extent to which y is more unequal than x elsewhere. It is not contrary to the spirit of a fuzzy approach to rule out the possibility of such sharp perceptions.

The reader may easily check that none of the criticisms levelled against the Gini and Lorenz ranking in Section 2 apply in the case of our new fuzzy measure.

Given the close relation between inequality and poverty measurement (see Sen, 1976; Osmani, 1983), the existing exact measures of poverty would have most of the shortcomings pointed out in Section 2 in the context of inequality. In addition the controversy about the exact positioning of the poverty line implies considerable scope for the use of fuzzy measurement in ranking states according to poverty (see Kundu and Smith, 1983).

5. The fuzzy measure and the Gini ranking

There is an interesting connection between G and the fuzzy inequality relation, R. The ensuing discussion elaborates this connection.

In the theory of fuzzy sets, some attempt has been made to find for every fuzzy set (and therefore every fuzzy relation), its 'nearest exact set' (Kaufmann, 1975). This is an important exercise in economics, because even when an agent or a group has fuzzy preferences, when it comes to shopping or selecting a policy option, it has to, perforce, make an exact choice. Hence, even though our actual inequality relation may be fuzzy we may, in some situations, be compelled to work with its nearest exact relation. It will be shown here that G is a nearest exact relation of P. This becomes obvious enough once we define 'nearest' carefully.

The usual definition is as follows: Let $\bar{U}$ be the (exact) universal set which is finite. If A and B are fuzzy subsets of $\bar{U}$, the *generalised Hamming distance* between them is defined as

$$d(A,B)=\sum_{x\in\bar{U}}|A(x)-B(x)|.$$

Given a fuzzy set A in $\bar{U}$, the class of *nearest exact sets* to A, $\bar{N}[A]$ is defined as follows:

$$E\in\bar{N}[A]\leftrightarrow d(A,E)=\min_{E'\subset U} d(A,E'). \tag{8}$$

The symbol '$\subset$' is used to represent 'is an exact subset of'. This definition has been used in Basu (1984) and Dutta, Panda and Pattanaik (1984). The trouble with this definition is that it requires the universal set to be finite for otherwise $d(\cdot)$ may not be well-defined.

Now consider a related definition of a nearest exact set which does not make use of any distance function: Let the (exact) universal set be U, which may or may not be finite. For any fuzzy set A in U, define $N[A]$ as follows:

$$E\in N[A]\leftrightarrow\{x\in U\mid A(x)>0.5\}\subset E\subset\{x\in U\mid A(x)\geq 0.5\}.^{11} \tag{9}$$

[11] Though $N[A]$, unlike $\bar{N}[A]$, is defined without reference to a distance function, it is possible to give a distance-based definition of $N[A]$: Let A and B be fuzzy subsets of U. If W is a finite exact subset of U, we will denote the distance between A and B in W by $d(A,B\mid W)$. Hence, $d(A,B\mid W)$ $\sum_{x\in W}|A(x)-B(x)|$.

Lemma. $E\in N[A]$ *iff for every finite* $W\subset U$,

$$d(A,E\mid W)=\min_{E'\subset U} d(A,E'\mid W). \tag{9.1}$$

A sketch of the proof is as follows: Suppose $E\in N[A]$. Then (9) holds. Hence, if $W\subset U$, $W\cap\{x\in u\mid A(x)>0.5\}\subset W\cap E\subset W\cap\{x\in U\mid A(x)\geq 0.5\}$. This implies

$$\{x\in W\mid A(x)>0.5\}\subset W\cap E\subset\{x\in W\mid A(x)\geq 0.5\}. \tag{9.2}$$

Suppose W is finite. Since we know that E defined in (8) coincides with E defined in (9) if U is finite, it follows from the finiteness of W that (9.1) above must be true. Conversely, since W is finite, (9.1) implies (9.2). If (9.2) is true for every finite W, it is obvious that (9) must be true, i.e. $E\in N[A]$.

It is well-known that if U is finite then the two definitions coincide, i.e. for all A, $N[A]=\bar{N}[A]$. If U is infinite, $\bar{N}[\cdot]$ ceases to be defined, but $N[\cdot]$ remains well-defined. Thus we shall here use the definition $N[A]$. Given a fuzzy relation P, a reference to its 'nearest exact relation' should be taken to mean a reference to some $\tilde{P}\in N[P]$.

What is interesting and easy to check is that if P is the fuzzy inequality relation, (6), then

$$G\in N[P],$$

that is, the strong Gini(E) relation is a best exact approximation of the fuzzy inequality relation. In the light of footnote 4, it is clear that if we were dealing with continuous distributions, the Gini(E) criterion would be equivalent to the Gini criterion and the Gini relation would be a best exact approximation of our fuzzy relation.

While this may be thought of as a property of the fuzzy relation, it could, conversely, be seen as a partial interpretation of the Gini criterion. The Gini relation could be thought of as reflecting not our actual ranking of states according inequality, but an approximation (within the domain of exact relations) of the actual.

6. An illustration

A practical advantage of the fuzzy inequality measure is that it is very easy to compute.[12] I illustrate this and also the contrast between the fuzzy measure and

Table 1
Income distribution in three countries

Cumulative share of bottom percent	U.K. 1964	Netherlands 1962	West Germany 1964
10	2.0	1.3	2.1
20	5.1	4.0	5.3
30	9.3	8.2	10.0
40	15.3	14.0	15.4
50	22.8	21.4	21.9
60	31.9	30.0	29.1
70	42.9	40.0	37.5
80	55.8	51.6	47.1
90	70.7	66.2	58.6
100	100.0	100.0	100.0

[12] To the extent that complete income distribution data is not easily available there can arise delicate questions of approximation. But these would be no more intricate than the difficulties that arise in measuring, for example, the Gini coefficient (see Kakwani, and Podder, 1976). If we use grouped data and ignore intra-group inequalities, computation becomes extremely simple, as the example in this section illustrates.

Table 2
Boolean relations matrices

Lorenz (L)

	U.K.	Neth.	W.G.
U.K.		1	0
Neth.	0		0
W.G.	0	0	

Gini (G)

	U.K.	Neth.	W.G.
U.K.		1	1
Neth.	0		1
W.G.	0	0	

Fuzzy (P)

	U.K.	Neth.	W.G.
U.K.		1	0.96
Neth.	0		0.73
W.G.	0.04	0.27	

other conventional measures by using the data given in Atkinson (1975, p. 46, table 3.3) and reproduced here in Table 1. If we ignore intra-group inequality, or alternatively suppose that each economy has 10 individuals, each representing a percentile[13], then we could easily compute the Lorenz, Gini(E) and Fuzzy relations over the set of alternative states, namely, (U.K., Netherlands, West Germany).

A fuzzy relation is difficult to represent pictorially. It may, therefore, be best to represent all these relations using 'Boolean relations matrices'. Given a set of social states X, a Boolean relations matrix, $[a_{xy}]_{x,y \in X}$, is defined as follows:

$$a_{xy} = P(x, y), \quad \forall x, y \in X.$$

Thus for every relation, P, there exists a unique Boolean relations matrix. Hence the three matrices in Table 2, fully describe the relations, L, G and P. Since P was defined only over distinct pairs, the diagonal elements of the matrices are left vacant. Note that between U.K. and the Netherlands, the former has a better distribution everywhere. Between U.K. and West Germany it is a bit more controversial because though U.K. has a better distribution almost everywhere, at lower levels of income West Germany is more equitable. This is, however, not reflected in the Gini(E) relation which ranks U.K. as unequivocally more equitable than the Netherlands and West Germany. The Gini ranking being *exact* and *complete* has no option but to overlook such details. If, in response, we turn to the Lorenz relation we are greeted with widespread reticence, the counterpart of which in the Boolean matrix is a plethora of zeroes. The fact that West Germany is more equitable than U.K. over a tiny stretch, is enough for there to be no ranking between these two economies. The new fuzzy inequality relation, developed in this paper contrasts itself to the Gini and Lorenz rankings by neither ignoring small conflicts nor being totally silenced by them.

[13] In order to remain within the framework of this paper, we shall here assume the latter. We shall also suppose that the aggregate income in each country is 100 so that the figures in the table may be thought of as actual income, instead of percentages.

References

A.B. Atkinson, On the measurement of inequality, J. Econ. Theory 2 (1970) 244–263.

A.B. Atkinson, The Economics of Inequality (Clarendon Press, Oxford, 1975).

S. Barbera and F. Valenciano, Collective probabilistic judgements, Econometrica 51 (1983) 1033–1046.

K. Basu, Determinateness of the utility function: Revisiting a controversy of the thirties, Rev. Econ. Studies 49 (1982) 307–311.

K. Basu, Cardinal utility, utilitarianism and a class of invariance axioms in welfare analysis, J. Math. Econ. 12 (1983) 193–206.

K. Basu, Fuzzy revealed preference theory, J. Econ. Theory 32 (1984) 212–227.

P. Dasgupta, A.K. Sen and D. Starrett, Notes on the measurement of inequality, J. Econ. Theory 6 (1973) 180–187.

P. Dasgupta, The Control of Resources (Basil Blackwell, Oxford, 1982).

D. Dubois and H. Prade, Fuzzy Sets and Systems: Theory and Applications (Academic Press, New York, 1980).

B. Dutta, Fuzzy preferences and social choice, Mathematical Social Sciences 13 (1985) 215–229.

B. Dutta, S.C. Panda and P.K. Pattanaik, Exact choice and revelation of fuzzy preferences, Mathematical Social Sciences 11 (1986) 53–68.

N. Kakwani and N. Podder, Efficient estimation of Lorenz curve and other associated inequality measures from grouped data, Econometrica 44 (1976) 137–148.

A. Kaufman, Theory of Fuzzy Subsets I (Academic Press, New York, 1975).

A. Kundu, and T.E. Smith, An impossibility theorem on poverty indices, Int. Econ. Rev. 24 (1983) 423–434.

O. Lange, The determinateness of the utility function, Rev. Econ. Studies 1 (1933) 218–225.

S.A. Orlovsky, Decision-making with a fuzzy preference relation, Fuzzy Sets and Systems 1 (1978) 155–167.

S.R. Osmani, Economic Inequality and Group Welfare (Oxford University Press, Oxford, 1982).

G. Pyatt, On the interpretation and disaggregation of Gini coefficients, Econ. J. 86 (1976) 243–255.

K.R. Ranadive, The equality of incomes in India, Bulletin of the Oxford Institute of Statistics 27 (1965) 119–134.

A.K. Sen, On Economic Inequality (Clarendon Press, Oxford, 1973).

A.K. Sen, Informational bases of alternative welfare approaches, J. Public Econ. 3 (1974) 387–403.

A.K. Sen, Poverty: An ordinal approach to measurement, Econometrica 44 (1976) 219–231.

N. Takayama, Poverty, income inequality and their measures: Professor Sen's axiomatic approach reconsidered, Econometrica 47 (1979) 747–759.

S. Tendulkar, Economic inequality in an Indian perspective, in: A. Beteille, ed., Equality and Inequality (Oxford University Press, Delhi, 1983).

D. Thon, An axiomatization of the Gini coefficient, Mathematical Social Sciences 2 (1982) 131–430.

J.A. Weymark, Generalised Gini inequality indices, Mathematical Social Sciences 1 (1981) 409–430.