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NOTES AND COMMENTS

POVERTY MEASUREMENT: A DECOMPOSITION OF THE NORMALIZATION AXIOM

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1. MOTIVATION

IN SEN'S [**8**] derivation of a poverty measure the crucial axiom is the "normalization axiom." This axiom, by requiring the poverty measure to be of a certain form in a very *special* case, manages to precipitate a unique measure for *all* cases. It is quite an ingenious axiom and not surprisingly it has received a large amount of attention.[2] The purpose of the present paper is not to evaluate this axiom but to facilitate its evaluation. I propose three elementary axioms and then using a theorem on affine functions which I established elsewhere (Basu [**3**]) show that these axioms are equivalent to Sen's normalization axiom, thereby allowing us to view the latter in a different light. Since the normalization axiom has been a staple assumption in several recent efforts to derive poverty measures, the present paper may be thought of as providing an alternative axiomatization for each of these cases.

2. FRAMEWORK AND RECAPITULATION

A *social state* is a nonnegative finite vector x such that x_i denotes the income of the ith poorest person (ties being broken arbitrarily). The set of all alternative social states, X, is therefore given by

$$X = \bigcup_{n=1}^{\infty} X^n, \quad \text{where} \quad X^n = \{x \in E^n | 0 \leq x_1 \leq x_2 \cdots \leq x_n\},$$

where E^n is the n-dimensional Euclidean space. Let z be the *poverty line*, meaning that a person is considered poor if and only if his income is not above z. Given a state $x \in X$, let $S(x)$ be the set of poor people in x and let $\# S(x)$ and $\# x$ denote the number of poor people and the number of people in x. A *poverty measure* P is a mapping from X to $[0, 1]$, such that $P(x)$ denotes the extent of poverty in x.

Two traditional poverty measures are the *head-count ratio*, H, and the *income-gap ratio*, I; and these are defined as follows:

$$\forall x \in X, \qquad H(x) = \frac{\# S(x)}{\# x},$$

$$\forall x \in X, \qquad I(x) = \sum_{i \in S(x)} (z - x_i)/\# S(x) \cdot z.$$

It is now well-accepted that neither H nor I, nor any combination of H and I, adequately capture the idea of poverty. If, of two poor people, some income is transferred from the poorer to the richer (who remains below the poverty line even after the transfer), then H and I, and therefore all combinations of these as well, will remain unchanged. Yet it is reasonable to argue (see Sen [**8**], Takayama [**10**], Kakwani [**5**]) that a transfer of this kind ought to register an increase in a "true" index of poverty.

However, there is one special case where H and I may be adequate. In Sen's [**9**, p. 295] words: "... we could criticize the *combination* of [H and I] only for their insensitivity to variations of distribution of income among the poor. If we were, then, to confine ourselves to cases in which all the poor have precisely the same income, it is reasonable

[1] I am grateful to an anonymous referee for his comments.

[2] The normalization axiom has been criticized, modified and used for deriving alternative measures of poverty: see Anand [**1**], Thon [**11**], Takayama [**10**], Kakwani [**5**].

to expect that H and I together may do the job." This seems entirely acceptable and so let us state this requirement formally.

Let $\hat{X}$ be the set of those states in X in which all the poor earn the same income. Hence, what we just argued may be stated formally as follows: There exists a function f such that

$$(1) \qquad \forall x \in \hat{X}, \quad P(x) = f(H(x), I(x)).$$

Beginning from this very reasonable position the normalization axiom takes a jump. It asserts without justification that the combination should take the simple multiplicative form:

AXIOM N: *For all* $x \in \hat{X}$, $P(x) = H(x) \cdot I(x)$.

This specific form plays a very important role in the derivation of the poverty measure and the question must arise: How do we justify this specific multiplicative form? In the next section I specify three axioms which have intuitively clear meanings (whether they are ethically appealing or not is left for the reader to judge) and which are shown to be equivalent to Axiom N.

3. THE DECOMPOSITION THEOREM

Our axioms are conditions on the function, f, in (1). The domain of the function plays an important role; so let us specify this carefully. Note that, given the definition of I and $\hat{X}$, for all $x \in \hat{X}$, $I(x) \in [0, 1]$ and for all $r \in [0, 1]$ we can find $x \in \hat{X}$ such that $I(x) = r$. Hence, $I(\hat{X}) = [0, 1]$. $H(\hat{X})$ is, however, not equal to $[0, 1]$. Since $\# S(x)$ and $\# x$ are both integers, clearly $H(x)$ is a rational number. It is easy to verify that if Q is the set of all rational numbers, then $H(\hat{X}) = Q \cap [0, 1]$. Hence instead of defining f on $\hat{X}$ as in (1), we can instead assert that

$$(2) \qquad f: (Q \cap [0, 1]) \times [0, 1] \rightarrow [0, 1].$$

This is to be interpreted as follows: If $H_i \in Q \cap [0, 1]$ and $I_i \in [0, 1]$, then $f(H_i, I_i)$ is the poverty index of all those social states $x \in \hat{X}$, such that $H(x) = H_i$ and $I(x) = I_i$.

If in a social state $H = 1$ and $I = 1$, that is, poverty is total, then we shall set $P = 1$. Also as H or I goes to zero (in a state where everybody is equally poor), P must go to zero as well. This latter may be stated in an alternative (stronger) way: If $H = 0$ or $I = 0$, P must be 0 and f is continuous in H and I. These observations are embodied in our first axiom:

AXIOM 1: (a) $f(1, 1) = 1$; (b) $\lim_{H \rightarrow 0} f(H, I) = \lim_{I \rightarrow 0} f(H, I) = 0$.

Let us return to the argument of Sen quoted in the last section which justifies the use of a combination of H and I for social states belonging to $\hat{X}$. What it says is this: If all poor people have the same income (i.e. $x \in \hat{X}$) and if I remains unchanged, then changes in the poverty measure should reflect changes in the head-count ratio only. This, as we argued earlier, is quite reasonable. What we have to decide is *to what extent* should P reflect H in such a case. Clearly, we would want P to rise if H rises. Formally,

$$(3) \qquad \forall H_1, H_2 \in Q \cap [0, 1] \text{ and } \forall I \in [0, 1], \quad [H_1 \geq H_2] \rightarrow [f(H_1, I) \geq f(H_2, I)].$$

It is possible to go along further and assert that if all poor have the same income and I remains unchanged, P should preserve order, not only over H—as in (3)—but over first-differences, in H. Let us state this in an axiom.

AXIOM 2: $\forall H_1, H_2, H_3, H_4 \in Q \cap [0, 1]$ and $\forall I \in [0, 1]$,

$$[H_1 - H_2 \geq H_3 - H_4] \rightarrow [f(H_1, I) - f(H_2, I) \geq f(H_3, I) - f(H_4, I)].$$

By taking $H_2 = H_4$, it is clear that Axiom 2 implies (3). Axiom 2 is thus a strengthening of (3). Note that we could ask for P to reflect H even better than Axiom 2 suggests. For instance, we could require P to preserve differences of differences, or differences of differences of differences and so forth, of H.[3]

With a symmetric justification, we may state the following axiom.

AXIOM 3: $\forall I_1, I_2, I_3, I_4 \in [0, 1]$ and $\forall H \in Q \cap [0, 1]$,

$$[I_1 - I_2 \geqslant I_3 - I_4] \to [f(H, I_1) - f(H, I_2) \geqslant f(H, I_3) - f(H, I_4)].$$

What is interesting is that because of the structure of f's domain, namely that it operates on $Q \cap [0, 1] \times [0, 1]$, which is given naturally by the framework of the problem, Axioms 1, 2, and 3 imply Axiom N.

THEOREM: *Axioms* 1–3 *are equivalent to Axiom N.*

The proof of the theorem is greatly facilitated by using a result on affine functions proved in Basu [**3**] and briefly stated in the next section.

4. THE AFFINE FUNCTION LEMMA

Let $g: B \to E^1$ where $B \subset E^1$.

DEFINITION 1: g is a *positive affine transformation* if there exist $a, b \in E^1$, with $b > 0$, such that for all $r \in B$, $f(r) = a + br$.

DEFINITION 2: g is a *(first-) difference preserving transformation* if for all $r_1, r_2, r_3, r_4 \in B$, $r_1 - r_2 \geqslant r_3 - r_4 \to f(r_1) - f(r_2) \geqslant f(r_3) - f(r_4)$.

LEMMA: *If* $g: B \to E^1$ *and the closure of* B *is an interval then* g *is a positive affine transformation if and only if it is a difference preserving transformation.*

5. PROOF OF THEOREM

That Axiom N implies Axioms 1–3 is obvious.

Suppose Axioms 1–3 are valid. For every value of $H \in Q \cap [0, 1], f(H, \cdot)$ could be thought of as a mapping on [0, 1]. Axiom 3 asserts that $f(H, \cdot)$ is a difference preserving transformation on [0, 1]. Hence by the Lemma, for each value of H, $\exists a(H), b(H) \in E^1$ such that $\forall I \in [0, 1]$

(4) $$f(H, I) = a(H) + b(H) \cdot I.$$

Note that as $I \to 0$, $f(H, I) \to a(H)$. Thus by Axiom 1(b), $a(H) = 0$. Therefore,

(5) $$f(H, I) = b(H) \cdot I.$$

[3] The preservation of, for instance, differences of differences would imply:

$$\forall H_1, \ldots, H_8 \in Q \cap [0, 1] \quad \text{and} \quad \forall I \in [0, 1]:$$

$$[\{(H_1 - H_2) - (H_3 - H_4)\} \geqslant \{(H_5 - H_6) - (H_7 - H_8)\}]$$
$$\to [\{(f(H_1, I) - f(H_2, I)) - (f(H_3, I) - f(H_4, I))\}$$
$$\geqslant \{(f(H_5, I) - F(H_6, I)) - (f(H_7, I) - f(H_8, I))\}].$$

While the preservation of differences of higher and higher orders (e.g., differences, differences of differences, etc.) are *distinct* properties in a "primitive" sense, they turn out to be equivalent if the domain of the function has certain restrictions. This is known in the context of utility theory; and the theorem below implies that even in the present context this equivalence is true.

Now, consider the mapping $f(\cdot, I)$. For each value of I, $f(\cdot, I)$ is a mapping on $Q \cap [0,1]$. The closure of $Q \cap [0,1]$ is $[0,1]$. Thus the lemma becomes applicable. Thus using Axiom 2 and a similar argument as above, we get

$$f(H, I) = c(I) + d(I) \cdot H,$$

where $c(I)$, $d(I) \in E^1$. By Axiom 1(b), $c(I) = 0$. Thus,

$$f(H, I) = d(I) \cdot H. \tag{7}$$

(5) and (7) $\rightarrow b(H) = Hd(I)/I$. Since $b(H)$ depends on H alone, $d(I)/I$ must be a constant, say k. Hence, by substitution in (5), we have:

$$f(H, I) = HIk.$$

By Axiom 1(a), $k = 1$, which completes the proof.

6. REMARKS

REMARK 1: This paper makes use of some results in modern utility theory, the antecedents of which go back to Lange [**6**] and Samuelson [**7**]. The original debate was concerned with the relation between the properties of cardinality and first-difference comparability of utility. It has been shown that as long as the domain of utility transformations lies within a certain class, the two properties are equivalent (Basu [**3**]). The mathematical structure of this equivalence result is quite independent of the debate on utility measurement and its essence is captured in the Lemma stated in Section 4 above. It was possible to bring this result to bear upon the problem of poverty measurement because poverty measures, for natural reasons, happen to be defined on a domain which has the required property to make the equivalence result applicable.

A weaker version of the affine function lemma (used in Basu [**2**]) interestingly does not work here. The weaker version asserts the equivalence of a positive affine transformation and a difference preserving transformation if the domain of the mapping is an interval. In this paper $f(\cdot, I)$ operates only on the rational numbers in $[0,1]$ since H can take only such values. Hence the stronger version (stated in Section 4), which assures the equivalence of Definitions 1 and 2 even if the domain of the mapping is a dense subset of an interval, is crucial here.

REMARK 2: It is obvious that for each axiomatic derivation of a poverty measure which uses Axiom N (e.g., Sen [**8**], Anand [**1**], Kakwani [**5**]), we now have an alternative axiomatization which replaces Axiom N with Axioms 1–3.

REMARK 3: It is easy to see that if we assume that for all $x \in X$ (instead of restricting it to $\hat{X}$), $P(x) = f(H(x), I(x))$ as in (1) and then use Axioms 1–3 to restrict f, we get the result that for all $x \in X$, $P(x) = H(x) \cdot I(x)$. We have, therefore, established an axiomatization of the poverty measure

$$P = HI,$$

and hence, with slight adjustment, an axiomatization of Kakwani's measure (Kakwani [**5**, p. 329]) and also that of a special case of Clark, Hemming, and Ulph's [**4**] measure.

7. CONCLUSIONS

One of the objectives of the recent literature on poverty measurement is to integrate in a single measure three different aspects of poverty: (i) the "deprivation" aspect (how far below the poverty line are the poor?), (ii) the "population" aspect (what proportion of the people are poor?), and (iii) the "distribution" aspect (what is the extent of inequality among the poor?). In Sen's formulation and in most of the literature that has followed, the resolution of (i) and (ii) is especially significant because, apart from its inherent normative interest, the manner of resolving these suggests a unique way of resolving all three aspects.

In most works, the integration of the population and income aspects is achieved by the normalization axiom (Axiom N, above) or by some close variant of this. It was argued here that while the spirit of this axiom is appealing, its specific form demands closer evaluation because of its far-reaching implications. Three axioms were stated and shown to be jointly equivalent to Axiom N. These new axioms are intuitively more basic and two of them (namely, 2 and 3) have analogues in utility theory which have been much analyzed and are, thus, well-understood. The main theorem in this paper may, therefore, be viewed as an attempt to lend greater transparency to the crucial normalization axiom.

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