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# CARDINAL UTILITY, UTILITARIANISM, AND A CLASS OF INVARIANCE AXIOMS IN WELFARE ANALYSIS*

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This paper examines some problems of utility measurement and interpersonal comparisons. The following theorem is first established for real-valued mappings on a set of real numbers, $B$: If the closure of $B$ is an interval, then such a mapping is positive affine iff it is a first-difference preserving transformation. This theorem is used to axiomatize cardinal utility. Standard invariance axioms in social choice theory are interpreted in an alternative framework and this gives new results in axiomatizing utilitarianism. It is shown that the conflict between utilitariansim and Rawlsianism arises, in some sense, earlier than traditionally supposed.

## 1. Motivation

Does the ability to compare first-differences in utility imply cardinality? Attempts to answer this has been the source of some confusion. I have tried to argue elsewhere that this question has generated conflicting answers because of differences in implicit frameworks. In the 'traditional' framework [e.g., the one used by Samuelson (1947) in his *Foundations*], first-difference comparability does turn out to be equivalent to cardinality. On the other hand, in the 'modern' (choice theory) framework, these two concepts are distinct. The confusion has been persistent because while most people are aware of the two frameworks, they have considered the differences between these to be *stylistic* rather than *substantial.*

The existence of two frameworks for analysing concepts of intrapersonal utility measurement, suggests that there must exist *alternative* frameworks for interpersonal comparisons as well. It is argued here that the recent literature on social welfare analysis [e.g., Hammond (1976), d'Aspremont and Gevers (1977), Sen (1977), Maskin (1978) and Roberts (1980)] has been developed (implicitly) within one framework — the 'modern' one. This suggests that one could translate these definitions into a class of related ones in the 'traditional framework'. Such an exercise yields interesting insights. Some concepts which are distinct in the existing literature are logically indistinguishable in this alternative framework. A number of new informational characterisations of

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social welfare functionals are possible in this framework. Some results on utilitarianism and the conflict between utilitarianism and Rawlsianism are established as illustration. It is shown that the conflict between utilitarianism and Rawlsianism arises, *in some sense*, earlier than traditionally supposed; and also that we may think of utilitarianism as characterised by *inter*personal *almost level* compability and *intra*personal *difference* comparability.

I argue that there are good methodological reasons for adopting the 'traditional' framework rather than the modern one. The 'traditional' approach is shown to differ from the 'modern' one in its use of a kind of 'unrestricted domain' assumption. Since the theoretical economist's formulation is typically undertaken a priori — without knowing what the actual decision problem is — the 'unrestricted domain' assumption does appear attractive.

Instead of developing the results on cardinal utility and social choice theory separately, I establish an abstract result on affine functions which is then shown to lie at the nub of the two sets of problems. I state and prove the affine function theorem in the strongest form I am aware of. This result is shown to be the basis of a number of axiomatisations of cardinal utility [Lange (1933, 1934) and Samuelson (1938)] including Basu (1982). In all these papers and in many other attempts to axiomatise cardinal utility, it is assumed that the image of the utility function is connected. For example, such an assumption is implicit in axiom A6 of Suppes and Winet (1955). The present theorem, by making use of a standard lemma in analysis (the principle of extension by continuity), shows that this is not necessary. The theorem is stated in section 2 and proved in section 6.

Broadly speaking, there are two classes of axiomatisation of cardinal utility without introducing risk. One approach begins by assuming that real-valued utility functions exist and then imposes conditions on these to derive cardinal utility. Lange (1933), Samuelson (1938), Bernadelli (1934) and Basu (1980, 1982) are examples of this, though Samuelson does go beyond. The other approach begins *without* assuming the existence of utility functions. The 'primitives' are binary and quaternary relations. Conditions are imposed on these relations and then it is shown that (i) utility functions exist and (ii) utility is cardinal. Of course, steps (i) and (ii) are often diffused in the proofs. The best known result in this approach is that of Suppes and Winet (1955). There are many other such attempts [see Fishburn (1970, ch. 6)]. In a sense the first approach corresponds to step (ii) of the second approach. Therefore, while our Theorem 1 is a direct generalisation of the theorems in the first approach, it does throw some light on part of the second approach.

## 2. A theorem on affine functions

Let $R$ be the set of real numbers.

*Definition.* The mapping $f: B \to R$, where $B \subset R$, is a *positive affine transformation* (PAT) if $\exists a, b \in R$, with $b > 0$, such that $\forall t \in B$,

$$f(t) = a + bt.$$

*Definition.* The mapping $f: B \to R$, where $B \subset R$, is a *first-difference preserving transformation* (DPT) if $\forall t_1, t_2, t_3, t_4 \in B$,

$$t_1 - t_2 \geqq t_3 - t_4 \leftrightarrow f(t_1) - f(t_2) \geqq f(t_3) - f(t_4).$$

It is easy to see that if a mapping is a PAT, it must necessarily be a DPT: Suppose $f: B \to R$ is a PAT. Then there exists a positive real number $b$ such that for all $t_i, t_j$ in $B$, $f(t_i) - f(t_j) = b(t_i - t_j)$. Since $b$ is positive, it is clear that $f$ satisfies the first-difference preservation property. However, the reverse implication need not be true, that is, a function may be a DPT without being a PAT. This is demonstrated in the example in section 3. Thus PAT and DPT are distinct concepts.

What is interesting, however, is that the two concepts become logically indistinguishable with a little structure on the domain $B$. For instance, *if the domain is unrestricted, that is $B = R$, then DPT and PAT are equivalent.* What is surprising is that for this equivalence result not only does $B$ not have to coincide with $R$, but it need not even be a connected subset of $R$.

*Theorem 1. If $f: B \to R$ and $B$ is dense in a connected subset of $R$, then $f$ is a DPT if and only if it is a PAT.*

By making use of the Principle of Extension by Continuity [Dunford and Schwartz (1967, p. 23)] and the fact that points on a real line can be denoted exactly by their binary expansions, a simple and direct proof of this theorem is given in section 6.

## 3. Utility theory

Let $X$ be the set of alternatives. A *utility function*, $u$ is a real-valued function on $X$, i.e., $u: X \to R$. The word *transformation* is reserved for a real-valued mapping defined on a subset of $R$. An individual is characterised not by a single utility function but by a whole class of *permitted* utility functions. Let $f: B \to R$ be a *permitted* transformation of a utility function, $u$, with $B$ sufficiently large so that $u(X) \subset B$. Then $fu: X \to R$ is a permitted utility function. We could therefore think of an individual as an ordered pair $(u, \Omega | B)$, where $u$ is a reference utility function and $\Omega | B$ is a collection of permitted transformations, $f$, defined on a domain $B \subset R$, with $u(X) \subset B$. Therefore, for each individual, $(u, \Omega | B)$, the set of permitted utility functions,

$L(u, \Omega|B)$, is defined as follows:

$$L(u, \Omega|B) = \{fu \mid f \in \Omega|B\}. \tag{1}$$

We assume that $\Omega|B$ always contains the identity transformation so that $u \in L(u, \Omega|B)$, i.e., $u$ is itself a permitted utility function.

If we assume $B = R$, i.e., the transformations are defined on the entire domain, then the above method of characterising an individual is equivalent to the traditional approach [e.g., the approach in Lange (1933) and Samuelson (1947)]. This is the commonly used approach even today in many areas of economics. In the modern choice-theoretic approach, however, instead of beginning from an individual being an ordered pair $(u, \Omega|B)$ and then *deriving* the set of permitted utility functions $L(u, \Omega|B)$, we start *directly* by specifying a set of permitted utility functions, $L = \{u, \phi, \psi, \ldots\}$. This approach is, as a little reflection will show, equivalent to the above approach with the assumption that $u(X) = B$. This is so because given $\phi$, $u \in L$, we could always construct a transformation $f: u(X) \to R$ such that $f(u(x)) = \phi(x)$.

Hence the traditional and modern approaches could be thought of as special cases of the above characterisation of an individual. The two approaches differ in their choice of the following *alternative* assumptions.

An individual is an ordered pair $(u, \Omega|B)$, where:

*Assumption N* (No restriction on transformation domain). $B = R$.

*Assumption M* (Maximal restriction on transformation domain). $B = u(X)$.

Cardinal utility and first-difference comparable utility are defined as follows:

*Definition.* Individual $(u, \Omega|B)$ has *cardinal* utility if $\forall f \in \Omega|B, \exists a, b \in R$, with $b > 0$, such that $\forall t \in B$,

$$f(t) = a + bt.$$

*Definition.* Individual $(u, \Omega|B)$ *can compare first-differences of utility* if $\forall f \in \Omega|B$, $\forall t_1, t_2, t_3, t_4 \in B$,

$$t_1 - t_2 \geqq t_3 - t_4 \leftrightarrow f(t_1) - f(t_2) \geqq f(t_3) - f(t_4).$$

In the light of these two definitions the conceptual difference between Assumptions N and M become clear. Given Assumption N, a transformation $f$ is defined on $R$, even though in actual practice $f$ is used to convert $u(X) \subset R$, and the properties of $f$ (e.g., cardinality) are defined on the entire

$R$ [and not just on $u(X)$]. Hence it is a kind of unrestricted domain assumption: the transformation $f$ is *equipped* to transform any element of $R$ though it *actually* has to transform elements of $u(X)$. We shall therefore refer to the traditional framework, i.e., one using Assumption N, as an *unrestricted domain framework*. The rationale for N is now clear and this clarifies why the unrestricted domain framework has been the more popular one in economics.

Cardinality and first-difference comparability appear to be distinct concepts and they have been treated as such in much of economics and psychometrics [see Seigel (1956)]. However, under a variety of situations the concepts become equivalent. The next result is an immediate corollary of Theorem 1.

*Corollary 1.1. Given Assumption N, or Assumption M with $u(X)$ being dense in a connected subset of $R$, the ability to compare first-differences of utility is equivalent to cardinality.*

This shows that in the traditional framework, first-difference comparability and cardinality are logically equivalent. But in the modern framework the concepts coincide only if certain conditions are satisfied. Corollary 1.1 is a stronger result than both the theorems in Basu (1982) and it shows that the results in Lange (1933) and Samuelson (1938) are applications of Theorem 1.

An interesting question which arises is that in the modern framework, i.e., given Assumption $M$, can the restriction on $u(X)$ be relaxed even further while retaining the equivalence between first-difference comparability and cardinality? The answer to this is yes, as will be obvious from the example below. What is however not immediately clear is what is the minimal structure on $u(X)$ which gives us the equivalence result. It may at first sight appear that the answer has something to do with *lacunae* in $B$ (in the context of Theorem 1). A lacuna of $B$ is a non-degenerate interval of $R$, disjoint with $B$ and having a lower and an upper bound in $B$ [Hildenbrand and Kirman (1974)]. But it is possible to show that the equivalence between PAT and DPT as in Theorem 1 does not hinge on the existence or non-existence of lacunae in $B$.

*Examples.* Let $B^1=[0,2]\cup[3,5]$ and $B^2=\{0,1,5\}$. Both $B^1$ and $B^2$ have lacunae. On $B^1$ if a function $f$ is a DPT it must be a PAT, but not so on $B^2$. Consider first $f:B^1\to R$. Let $f$ be a DPT. Since the closures of $[0,2]$ and $[3,5]$ are connected in $R$, by Theorem 1, the restrictions of $f$ to $[0,2]$ and $[3,5]$ must be PATs. Thus $\exists a,b,c,d\in R$ with $b$, $d>0$:

$$\forall t\in[0,2],\qquad f|[0,2](t)=a+bt,$$

$$\forall t\in[3,5],\qquad f|[3,5](t)=c+dt.$$

If $b \neq d$ then $f(2)-f(0) \neq f(5)-f(3)$, which contradicts the fact that $f$ is a DPT. Hence $b=d$. If $a \neq c$, then $f(3)-f(2) \neq f(4)-f(3)$ which contradicts the fact that $f$ is a DPT. Hence $a=c$. Therefore $f$ is a PAT on the entire domain of $B^1$.

Now consider $f: B^2 \to R$. Let $f(0)=0$, $f(1)=1$, $f(5)=6$, such an $f$ is a DPT but not a PAT.

Therefore, the absence or presence of lacunae does not in itself change anything. We move on, leaving the question of minimal restriction on $B$, which makes the concepts of DPT and PAT equivalent, an open one.

Theorem 1 throws light on another classical problem. In traditional discussions on the measurement of utility the term 'addible' utility often cropped up (this should not be confused with the modern concept of 'additive' utility). The precise meaning of addible utility was seldom stated, but a large number of economists [see, e.g., Alchian (1953) and Majumdar (1958)] seemed to suggest by this the simple idea that the addition of utilities from a group of alternatives $x_1, x_2, \ldots, x_n$ in $X$ is 'meaningful'. That is, it is possible to say which among two groups of alternatives in $X$ gives a greater utility. In other words, if in terms of one utility function, group 1 gives greater utility than group 2, then all permitted utility functions retain this ordering.

*Definition.* Individual $(u, \Omega | B)$ has *addible* utility if $\forall f \in \Omega | B$, $\forall t_1, \ldots, t_n$, $s_1, \ldots, s_m \in B$,

$$\sum_{i=1}^{n} t_i \gtreqless \sum_{i=1}^{m} s_i \leftrightarrow \sum_{i=1}^{n} f(t_i) \gtreqless \sum_{i=1}^{m} f(s_i).$$

What this definition says is that if the sum of utilities that a person derives from $n$ alternative bundles is greater than the sum derived from $m$ alternative bundles, then all permissible transformations of the utility function retain this feature. It can be shown that given some domain restrictions, addibility is equivalent to what is often referred to as ratio-scale measurability [Roberts (1980)] in the social choice literature.

*Definition.* Individual $(u, \Omega | B)$ has a *ratio-scale measurable* utility if $\forall f \in \Omega | B$, $\exists b > 0$, such that $\forall t \in B$,

$$f(t) = bt.$$

*Theorem 2. Given Assumption N, or Assumption M with u(X) being dense in a connected subset of R, an individual $(u, \Omega | B)$ has addible utility if and only if his utility is ratio-scale measurable.*

Given Theorem 1, the proof of Theorem 2 is straightforward and is relegated to section 6.

## 4. Transformation domains in social choice

In the late seventies there were attempts to enlarge the Arrowian framework of social choice theory to a utility-theoretic base. This has led to the opening up of an engaging area of research and a series of remarkable results [Hammond (1976), d'Aspremont and Gevers (1977), Sen (1977, 1982), Arrow (1977), Deschamps and Gevers (1978), Maskin (1978) and Roberts (1980)].

It has already been noted that there are two approaches to the problem of utility measurement: one making use of the unrestricted domain framework (Assumption N) and the maximal restriction approach (Assumption M). In the same spirit it is possible to conceive of two frameworks for social welfare analysis. I argue here that the existing literature (the references cited above) has, however, adopted the maximal restriction approach. There is no reason why that ought to be so. As argued above there is a certain desirable catholicity in the unrestricted domain framework. This idea is developed in this and the next sections in the context of social choice theory and this yields some surprising insights.

Let us first recall the standard framework using mainly Sen's (1977) notation. Let $X$ ($\# X \geqq 3$) be the set of alternative social states and $H$ ($\# H \equiv n < \infty$) the set of individuals. Let $W_i(\cdot)$ be individual $i$'s real-valued utility function defined on $X$. A *social welfare functional* (SWFL), $F$, specifies an ordering (i.e., a complete, reflexive and transitive binary relation), $R$, on $X$, for each $n$-tuple $\{W_i(\cdot)\}_{i \in H}$. Hence

$$R = F(\{W_i(\cdot)\}).$$

The domain of $F$ consists of all $n$-tuples $\{W_i(\cdot)\}$. Intrapersonal and interpersonal comparisons are introduced via alternative 'invariance requirements'. Consider three standard invariance axioms:

(CF) *Cardinal Full Comparable (or Co-cardinal)*: If $\{W_i(\cdot)\}$ and $\{W'_i(\cdot)\}$ are such that $\exists a, b \in R$, with $b > 0$, such that $\forall x \in X$, $\forall i \in H$, $W'_i(x) = a + bW_i(x)$, then $F(\{W_i(\cdot)\}) = F(\{W'_i(\cdot)\})$.

(CU) *Cardinal Unit Comparable*: If $\{W_i(\cdot)\}$ and $\{W'_i(\cdot)\}$ are such that $\exists a_1, \ldots, a_n$, $b \in R$, with $b > 0$, such that $\forall x \in X$, $\forall i \in H$, $W'_i(x) = a_i + b(W_i(x))$, then $F(\{W_i(\cdot)\}) = F(\{W'_i(\cdot)\})$.

(OL) *Ordinal Level Comparable (or Co-ordinal)*: If $\{W_i(\cdot)\}$ and $\{W'_i(\cdot)\}$ are such that $\exists$ a positive monotone transformation $\phi(\cdot)$, such that $\forall x \in X$, $\forall i \in H$, $W'_i(x) = \phi(W_i(x))$, then $F(\{W_i(\cdot)\}) = F(W'_i(\cdot)\})$.

Note that in these definitions, given two $n$-tuples, $\{W_i(\cdot)\}$ and $\{W'_i(\cdot)\}$, we may conceive of $\{W'_i(\cdot)\}$ as derived from $\{W_i(\cdot)\}$ by applying an $n$-tuple of transformations $\{\phi_i\}=\{\phi_1,\dots,\phi_n\}$, with $\phi_i$ being used to transform $W_i(\cdot)$, $\forall i \in H$. This is particularly transparent in Roberts' (1980) paper. Note, however, that in the above axioms each $\phi_i$ need operate only on $W_i(X)$. The properties of $\phi_i$ specified in these axioms are properties which it must satisfy on $W_i(X)$. This is not generally stated explicitly thereby corroborating the view that alternative domain restrictions of transformations have been typically thought of as differences in style rather than of substance.

As argued above, in the context of utility theory, there is good reason to suppose that each transformation should be potentially able to transform any element of $R$, despite the fact that it does not have to actually do so in a particular situation. And also, transformations should be thought of as satisfying a certain property (e.g., first-difference preservation) if they (i) satisfy the property over the domain of values that *actually* occur, *and* (ii) have the *potential* for satisfying it over the entire domain ($R$). The above definitions merely require (i). The motivation for an approach which uses (i) and (ii) is similar in spirit to the motivation which had led economists to use the standard unrestricted domain assumption in social choice theory (i.e., to define a collective choice rule on *all possible* $n$-tuples of individual orderings). Such an approach would correspond to the unrestricted domain framework or the traditional approach to the problem of intrapersonal utility measurement.

## 5. Utilitarianism

In the unrestricted domain framework CF and CU would be defined as follows, using a bar (e.g., $\overline{\text{CF}}$) to distinguish the new definitions:

($\overline{\text{CF}}$) If $\{W_i(\cdot)\}$ and $\{W'_i(\cdot)\}$ are such that $W'_i(\cdot)=\phi_i(W_i(\cdot))$, $\forall i \in H$, where $\{\phi_i\}$ is a transformation $n$-tuple defined on $R$ such that $\exists a, b \in R$, with $b>0$, such that $\forall t \in R$, $\forall i \in H$, $\phi_i(t)=a+bt$, then $F(\{W_i(\cdot)\})=F(\{W'_i(\cdot)\})$.

($\overline{\text{CU}}$) If $\{W_i(\cdot)\}$ and $\{W'_i(\cdot)\}$ are such that $W'_i(\cdot)=\phi_i(W_i(\cdot))$, $\forall i \in H$, where $\{\phi_i\}$ is a transformation $n$-tuple defined on $R$ such that $\exists a_1,\dots,a_n$, $b \in R$, with $b>0$, such that $\forall t \in R$, $\forall i \in H$, $\phi_i(t)=a_i+bt$, then $F(\{W_i(\cdot)\})=F(\{W'_i(\cdot)\})$.

It is not difficult to see that CF$\leftrightarrow\overline{\text{CF}}$ and CU$\leftrightarrow\overline{\text{CU}}$ — the change of framework does not make a difference thus far. The next concept, however, is framework sensitive.

Let $\bar{\alpha}$ be the invariance axiom given that interpersonal utilities are ordinal level comparable and that there exists one individual with difference comparable utility — i.e., he can compare his own first-differences in utility. (Of course, coupled with level comparability, the latter immediately implies that *everybody* can compare first-differences.)

($\bar{\alpha}$) If $\{W_i(\cdot)\}$ and $\{W_i'(\cdot)\}$ are such that $W_i'(\cdot)=\phi_i(W_i(\cdot))$, $\forall i \in H$, where $\{\phi_i\}$ is a transformation $n$-tuple defined on $R$ such that
  (1) $\exists k \in H$, $\forall t_1, t_2, t_3, t_4 \in R$, $t_1 - t_2 \geqq t_3 - t_4 \leftrightarrow \phi_k(t_1) - \phi_k(t_2) \geqq \phi_k(t_3) - \phi_k(t_4)$, and
  (2) $\phi_i(\cdot)=\phi(\cdot)$, $\forall i \in H$ and $\phi(\cdot)$ is a positive monotone transformation,
  then $F(\{W_i(\cdot)\})=F(\{W_i'(\cdot)\})$.

The counterpart of $\bar{\alpha}$ in the framework conventionally used in modern social choice theory, i.e., in the maximal restriction framework (we may refer to such an axiom as $\alpha$) is logically distinct from $\bar{\alpha}$.

What is surprising is that $\bar{\alpha}$ and $\overline{\text{CF}}$ are logically equivalent. In other words, in the unrestricted domain framework, the concept of cardinal full comparability is equilivalent to ordinal level comparability coupled with *intra*personal difference comparability. This is precisely the assertion of the next theorem, the proof of which is given in section 6.

*Theorem 3. In the unrestricted domain framework, ordinal level comparability and the existence of an individual who can compare first-differences of his own utility is equivalent to cardinal full comparability, i.e.,* $\bar{\alpha} \leftrightarrow \overline{CF}$.

Two of the most discussed SWFLs are utilitarianism and Rawlsianism. Consider two slight variants of these: A SWFL, $F$, is *utilitarian-type* if $\forall x, y \in X$, $\forall \{W_i(\cdot)\}$,

$$\left[\sum_{i=1}^{n} W_i(x) > \sum_{i=1}^{n} W_i(y)\right] \quad \text{implies} \quad [xPy],$$

where $P$ is the asymmetric part of $F(\{W_i(\cdot)\})$.

Now consider the leximin rule. Let $k(x)$ denote the $k$th worst-off person (ties being broken arbitrarily) in state $x$ given an $n$-tuple $\{W_i(\cdot)\}$. A SWFL, $F$, is *leximin* if $\forall x, y \in X$, $\forall \{W_i(\cdot)\}$, $xPy$ if and only if $\exists k \leqq n$ such that $W_{k(x)}(x) > W_{k(y)}(y)$ and $\forall r < k$, $W_{r(x)}(x) = W_{r(y)}(y)$, where $P$ is the asymmetric part of $F(\{W_i(\cdot)\})$.

Given the prominence of Rawlsianism and utilitarianism in modern social choice theory, it is but natural that a lot of effort has gone into isolating situations where these two ethical principles are in direct conflict. One of the most interesting findings in this direction is a result due to Deschamps and

Gevers (1978). They showed that if a SWFL satisfies the properties of independence of irrelevant alternatives, the Pareto criterion, anonymity, a certain minimal equity axiom and separability [for formal definitions, see Deschamps and Gevers (1978) or Sen (1977)] — we refer to this axiom set as the D–G axiom set — then if we assume cardinal full comparability, the leximin criterion and utilitarian-type criterion are in direct conflict because these are the only two SWFLs satisfying the D–G axiom set.

What Theorem 3 suggests is that we adopt the unrestricted domain framework then this direct confrontation between utilitarianism and Rawls arises as soon as we assume ordinal level comparability and the existence of an individual who can compare his own first-differences.

*Corollary 3.1. Given the D–G axiom set, if a SWFL satisfies $\bar{\alpha}$ it must be either utilitarian-type or leximin.*

This corollary is an immediate consequence of the Deschamps–Gevers theorem and Theorem 3 above.

Let us now turn to the characterisation of utilitariansim in the unrestricted domain framework. A SWFL, $F$, is *utilitarian* if $\forall x, y \in X$, $\forall \{W_i(\cdot)\}$,

$$\left[\sum_{i=1}^{n} W_i(x) \geqq \sum_{i=1}^{n} W_i(y)\right] \quad \text{implies} \quad [xRy],$$

where

$$R = F(\{W_i(\cdot)\}).$$

Note that one obvious implication of being able to make level comparisons between individuals $i$ and $j$ is that if $W_i(x) = W_j(y)$ and $W_i(z) = W_j(v)$, then given a permitted transformation $n$-tuple, $\{\phi_i\}$, $\phi_i(W_i(x)) - \phi_i(W_i(z)) = \phi_j(W_j(y)) - \phi_j(W_j(v))$. This part of level comparability may be referred to as *almost* level comparability. The invariance axiom $\bar{\beta}$ below is a formal statement of almost level comparability of interpersonal utilities plus intrapersonal first-difference comparability, with (2) in the definition of $\bar{\beta}$ capturing the idea of almost level comparability discussed above.

($\bar{\beta}$) If $\{W_i(\cdot)\}$ and $\{W_i'(\cdot)\}$ are such that $W_i'(\cdot) = \phi_i(W_i(\cdot))$, $\forall i \in H$, where $\{\phi_i\}$ is a transformation $n$-tuple defined on $R$ such that

(1) $\forall k \in H, \forall t_1, t_2, t_3, t_4 \in R$, $t_1 - t_2 \geqq t_3 - t_4 \leftrightarrow \phi_k(t_1) - \phi_k(t_2) \geqq \phi_k(t_3) - \phi_k(t_4)$, and

(2) $\forall i, j \in H$, $\forall t_1, t_2 \in R$, $\phi_i(t_1) - \phi_i(t_2) = \phi_j(t_1) - \phi_j(t_2)$,

then $F(\{W_i(\cdot)\}) = F(\{W_i'(\cdot)\})$.

*Theorem 4.* *In the unrestricted domain framework, almost ordinal level comparability and the ability of every individual to compare first-differences of his own utility is equivalent to cardinal unit comparability, i.e.,* $\bar{\beta} \leftrightarrow \overline{CU}$.

In a well-known paper, d'Aspremont and Gevers (1977) proved that if a SWFL satisfies the properties of independence of irrelevant alternatives, the Pareto criterion and anonymity [for formal definitions, see d'Aspremont and Gevers (1977) or Sen (1977)] — we refer to these as the D′–G axiom set — and also CU, then it must be a utilitarian SWFL. This result, the fact that $\overline{\text{CU}} \leftrightarrow \text{CU}$ and Theorem 4 suggest the following characterisation of utilitarianism.

*Corollary 4.1.* *If a SWFL satisfies the D′–G axiom set and* $\bar{\beta}$*, it must be utilitarian.*

A similar exercise is possible for Rawls' difference principle, in particular, the leximin rule.

## 6. Proofs

The proof of Theorem 1 is greatly facilitated by using a standard lemma in mathematics [see Dunford and Schwartz (1967) for a proof]:

*Lemma* (*Principle of Extension by Continuity*). *Let* $X$ *and* $Y$ *be metric spaces and let* $Y$ *be complete. If* $f: A \to Y$ *is uniformly continuous on the dense subset* $A$ *of* $X$*, then* $f$ *has a unique continuous extension* $g: X \to Y$*. This unique extension is uniformly continuous on* $X$.

The following notation is useful. Given that $f: B \to R$, $\forall A \subset R$, $f(A) \equiv \{y \in R \mid y = f(t), t \in A \cap B\}$. Also $\forall A \subset R$, $\bar{A}$ denotes the closure of $A$ in $R$.

*Proof of Theorem 1.* Assume that $\bar{B}$ is a connected subset of $R$ and that $f$ is a DPT. If $B$ is a singleton then the result is immediate. Hence assume $\# B > 1$.

As a first step note that $f$ is a monotonically increasing function, i.e.,

$$\forall t_1, t_2 \in B, \quad [t_2 > t_1] \leftrightarrow [f(t_2) > f(t_1)]. \tag{1}$$

Let $t_1, t_2 \in B$. Then

$$t_2 > t_1 \leftrightarrow t_2 - t_1 > t_1 - t_1$$
$$\leftrightarrow f(t_2) - f(t_1) > f(t_1) - f(t_1), \quad \text{since } f \text{ is a DPT,}$$
$$\leftrightarrow f(t_2) > f(t_1).$$

This establishes (1).

As a second step we prove that $f$ is a uniformly continuous function. Let $\varepsilon > 0$. Choose $t_1, t_2 \in B$ such that $t_1 \neq t_2$ and $|f(t_1) - f(t_2)| < \varepsilon$. The existence of such $t_1, t_2$ may be proved as follows: Assume no such $t_1, t_2$ exist. If $[s_1, s_2]$ is a non-degenerate interval with $s_1, s_2 \in B$, it is easy to check that $\#([s_1, s_2] \cap B) = \infty$. Hence, since no $t_1, t_2$ of the above description exists, $f$ must be unbounded on $[s_1, s_2] \cap B$. Hence $\exists s_3 \in [s_1, s_2] \cap B$ such that $f(s_3) \notin [f(s_1), f(s_2)]$. This violates (1). Therefore, $t_1, t_2$ of the above description exists. Let $|t_1 - t_2| = \delta$,

$$\forall s_1, s_2 \in B, \quad |s_1 - s_2| < \delta \rightarrow |s_1 - s_2| < |t_1 - t_2|$$
$$\rightarrow |f(s_1) - f(s_2)| < |f(t_1) - f(t_2)|,$$

since $f$ is a DPT,

$$\rightarrow |f(s_1) - f(s_2)| < \varepsilon.$$

Therefore $f$ is uniformly continuous.

By the above lemma, $\exists g: \bar{B} \rightarrow R$ which is a unique continuous extension of $f$. The proof is completed by showing that $g$ is a PAT on $\bar{B}$. This, in turn, is proved by taking an arbitrary bounded and non-degenerate interval in $\bar{B}$ and proving that the restriction of $g$ on this is a PAT.

Let $[t_*, t^*]$, $t_* \neq t^*$, be such an arbitrary interval in $\bar{B}$. Let $s_* \equiv g(t_*)$, $s^* \equiv g(t^*)$, $\hat{t} \equiv \frac{1}{2}(t_* + t^*)$ and $\hat{s} \equiv \frac{1}{2}(s_* + s^*)$. W first prove

$$g([t_*, \hat{t}[) = [s_*, \hat{s}[ \quad \text{and} \quad g([\hat{t}, t^*[) = [\hat{s}, s^*[, \tag{2}$$

i.e., each half is mapped into its corresponding half in the range. It is easily checked that by virtue of (1), $g$ is also a monotonically increasing function. This, and the continuity of $g$ implies that $g([t_*, t^*[) = [s_*, s^*[$ and $\exists$ a unique $s \in [s_*, s^*[$ such that $g([t_*, \hat{t}[) = [s_*, s[$ and $g([\hat{t}, t^*[) = [s, s^*[$.

To prove (2) we simply have to show that $s = \hat{s}$. Without loss of generality assume $s > \hat{s}$. Hence, $s - s_* > s^* - s$. Hence $\exists t_1, t_2 \in [t_*, \hat{t}[ \cap B$ such that $g(t_1) - g(t_2) > s^* - s$. Choose $t_3, t_4 \in [\hat{t}, t^*[ \cap B$ such that $t_3 - t_4 \geqq t_1 - t_2$. Since

$g([\hat{t}, t^*[) = [s, s^*[$, hence $g(t_3) - g(t_4) < s^* - s$. Thus $g(t_3) - g(t_4) < g(t_1) - g(t_2)$. Since $t_1, t_2, t_3, t_4 \in B$ this violates the fact that $f$ is a DPT. This establishes (2).

Let $\phi = [0, 1[ \to [t_*, t^*[$ be a PAT with $\phi([0, 1[) = [t_*, t^*[$. Similarly define $\psi: [s_*, s^*[ \to [0, 1[$ such that $\psi$ is a PAT with $\psi([s_*, s^*[) = [0, 1[$. Let $(g|I)$ be the restriction of $g$ to $[t_*, t^*[$. Define a composite mapping $\Delta \equiv \psi(g|I)\phi$. Clearly $\Delta: [0, 1[ \to [0, 1[$. Fig. 1 illustrates this mapping.

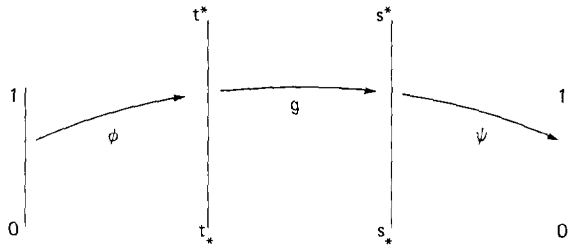


Fig. 1

It will be shown that $\Delta$ is an identity mapping. Let $a \in [0, 1[$ and let $\cdot a_1 a_2 a_3 \ldots$ be the binary expansion of $a$ and let $\cdot b_1 b_2 b_3 \ldots$ be the binary expansion of $\Delta(a)$. We first prove $a_1 = b_1$. Without loss of generality, suppose $a_1 = 0$. Thus $a \in [0, \frac{1}{2}[$. Hence $\phi(a) \in [t_*, \hat{t}[$. By (2), $(g|I)\phi(a) \in [s_*, \hat{s}[$. Hence $\psi(g|I)\phi(a) \in [0, \frac{1}{2}[$. Thus $b_1 = 0$.

In the same way we could prove $a_2 = b_2$. Since $a_1 = 0$, we know that $\phi(a)$ lies in the first half of $[t_*, t^*[$, i.e., $\phi(a) \in [t_*, \hat{t}[$. Now, since $t_*, t^*$ were chosen arbitrarily above, we could set $t_*$ as before and $t^*$ as what was earlier $\hat{t}$ and by a similar proof show that whichever half of $[t_*, \hat{t}[$, $\phi(a)$ happens to be in, $(g|I)\phi(a)$ would be in a corresponding half in the range. Hence $a_2 = b_2$; and by repeating, $a_n = b_n$, $\forall n$. Hence $\Delta$ is an identity mapping and in particular it is a PAT. By construction, $\phi$ and $\psi$ are PATs. Hence $g$ is a PAT on $[t_*, t^*[$. It is an immediate consequence of monotonicity and continuity of $g$ that $g(t^*) = s^*$. Hence $g$ is a PAT on $[t_*, t^*]$. Therefore, $g$ is a PAT on $\bar{B}$ and its restriction, $f$, must be a PAT.

It is obvious, as has been shown in section 2, that if $f$ is a PAT, it must be a DPT. Q.E.D.

*Proof of Theorem 2. Given addibility, it is obvious that* $f \in \Omega|B$ implies $f$ is a DPT. Given Assumption $N$ or Assumption $M$ with $u(X)$ being a connected subset of $R$, Theorem 1 implies $\exists a, b \in R$, with $b > 0$, such that $\forall t \in B$, $f(t) = a + bt$. Since individual $(u, \Omega|B)$ has addible utility, $1 + 0 = 0 \to f(1) + f(0) = f(1)$. Hence $f(0) = 0$. Therefore $a = 0$. Thus the individual's utility is *ratio-scale measurable*. The reverse implication is obvious. Q.E.D.

*Proof of Theorem* 3. Obviously $\bar{\alpha}\rightarrow\overline{CF}$. Assume $\overline{CF}$ is valid, and $\{W_i(\cdot)\}$ and $\{W_i'(\cdot)\}$ satisfy conditions (1) and (2) in the definition of $\bar{\alpha}$. It immediately follows that $\forall t_1, t_2, t_3, t_4 \in R, t_1 - t_2 \geqq t_3 - t_4 \leftrightarrow \phi(t_1) - \phi(t_2) \geqq \phi(t_3) - \phi(t_4)$. By Theorem 1, $\exists a, b \in R$, with $b \geqq 0$, such that $\forall t \in R$, $\phi(t) = a + bt$. Then by $\overline{CF}$, $F(\{W_i(\cdot)\}) = F(\{W_i'(\cdot)\})$. Hence $\bar{\alpha}$ holds. Q.E.D.

*Proof of Theorem 4.* Assume $\overline{CU}$ is valid and that $\{W_i(\cdot)\}$ and $\{W_i'(\cdot)\}$ satisfy conditions (1) and (2) in the definition of $\bar{\beta}$. By Theorem 1, $\forall k \in H$, $\exists a_k, b_k \in R$, with $b_k > 0$, such that $\forall t \in R$, $\phi_k(t) = a_k + b_k t$. Since $\phi_i(t_1) - \phi_i(t_2) = \phi_j(t_1) - \phi_j(t_2)$; hence $b_i(t_1 - t_2) = b_j(t_1 - t_2)$. Therefore $b_i = b_j = b$, $\forall i, j \in H$. Hence by $\overline{CU}$, $F(\{W_i(\cdot)\} = F(\{W_i'(\cdot)\})$. Hence $\bar{\beta}$ holds. Q.E.D.

## References


Alchian, A., 1953, The meaning of utility measurement, American Economic Review 42.

Arrow, K.J., 1977, Extended sympathy and the possibility of social choice, American Economic Review 67.

Basu, K., 1980, Revealed preference of government (Cambridge University Press, Cambridge).

Basu, K., 1982, Determinateness of the utility function: Revisiting a controversy of the thirties, Review of Economic Studies 49.

Bernadelli, H., 1934, Notes on the determinateness of the utility function, Review of Economic Studies 2.

d'Aspremont, C. and L. Gevers, 1977, Equity and the informational basis of collective choice, Review of Economic Studies 44.

Deschamps, R. and L. Gevers, 1978, Leximin and utilitarian rules: A joint characterisation, Journal of Economic Theory 17.

Dunford, N. and J.T. Schwartz, 1967, Linear operators, Part I: General theory (Interscience Publishers, New York).

Fishburn, P.C., 1970, Utility theory for decision making (Wiley, New York).

Hammond, P.J., 1976, Equity, Arrow's conditions and Rawls' difference principle, Econometrica 44.

Hildenbrand, W. and A.P. Kirman, 1976, Introduction to equilibrium analysis (North-Holland, Amsterdam).

Lange, O., 1933, The determinateness of the utility function, Review of Economic Studies 1.

Lange, O., 1934, Notes on the determinateness of the utility function, Review of Economic Studies 2.

Majumdar, T., 1958, The measurement of utility (Macmillan, New York).

Maskin, E., 1978, A theorem on utilitarianism, Review of Economic Studies 45.

Roberts, K.W.S., 1980, Interpersonal comparability and social choice theory, Review of Economic Studies 47.

Samuelson, P.A., 1938, The numerical representation of ordered classifications and the concept of utility, Review of Economic Studies 6.

Samuelson, P.A., 1947, Foundations of economic analysis (Harvard University Press, Cambridge, MA).

Seigel, S., 1957, A method for obtaining an ordered metric scale, Psychometrica 21.

Sen, A.K., 1977, On weights and measures: Informational constraints in social welfare analysis, Econometrica 45.

Sen, A.K., 1981, Social choice theory, in: K.J. Arrow and M. Intriligator, eds., Handbook of mathematical economics, Vol. III (North-Holland, Amsterdam).

Suppes, P. and M. Winet, 1955, An axiomatization of utility based on the notion of utility differences, Management Science 1.