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Determinateness of the Utility Function: Revisiting a Controversy of the Thirties

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It has been alleged that, contrary to the assumptions in Pareto's *Manual*, the ability to compare first-differences of utility implies cardinality. It is shown here that the validity of this theorem hinges critically on the framework of analysis. In the framework of the thirties it is valid because of the implicit use of an 'unrestricted domain' assumption. In the modern choice-theoretic context it is not in general true but it becomes valid if utility functions are continuous and are defined on a connected topological space.

1. INTRODUCTION: PARETO'S ASSUMPTIONS

In an important paper on utility theory, Oskar Lange (1933) alleged that a large number of Lausanne economists, particularly Pareto, Amoroso and Pietri–Tonelli, had assumed that an individual could compare (a) levels of utility, and (b) first differences of utility, and they had also assumed that utility was not cardinal. This, Lange claimed, was not consistent and he set out to show that (a) and (b) implied cardinal utility. A controversy ensued in which economists from Allen (1934) to Zeuthen (1936), including Alt (1936), Bernadelli (1934), Lange (1934), Phelps-Brown (1934) and Samuelson (1938), participated.

None of the existing proofs are, however, rigorous and moreover, none of these authors specified their framework of analysis clearly. This is a major shortcoming since the validity of the Lange hypothesis hinges critically on the framework of analysis. In (what in retrospect may be referred to as) the framework of the thirties or simply the traditional framework, the theorem is valid. It is more robust than generally supposed, because it is valid in the absence of differentiability and even continuity of the utility function. This is shown in the next section.

In the framework of modern social choice theory (see Sen (1970) Chapters 7 and 7*) the Lange hypothesis is invalid. Section 3 tries to locate the critical difference in the two frameworks which leads to these divergent implications.

While it is true that Pareto did assume—though only on occasions (see Chapter 4, Section 32, of his *Manual*)—that individuals could compare first-differences of utility, Lange's charge of inconsistency in Pareto's *Manual* is not completely tenable since it is not clear that Pareto was operating in the framework of the thirties.

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It is then argued in Section 3 that if, however, the utility functions are continuous and are defined on a connected topological space, then even in the modern framework the Lange hypothesis is valid. To be able to compare first-differences implies an ability to compare all higher-order differences!

The main theme of this paper is to examine and comment on the controversy of the thirties in the light of modern developments.

2. LANGE'S CLAIM

Let X be the set of alternatives. A utility function, u, is a real-valued function on X, i.e. $u:X\to R$, where R is the set of all real numbers. A *transformation* is a mapping from R to R. An individual is characterised by a reference utility function, u, and a set of permitted transformations, Ω. The individual's preference may be represented by u or any composite function fu, where $f\in\Omega$. Note that $fu:X\to R$.

Thus given u and Ω we have the set of permitted utility functions, $L(u, \Omega)$, defined as follows.[1]

$$L(u,\Omega)=\{fu\,|\,f\in\Omega\}.$$

The agent is supposed to have those preferences which are implied by *all* the utility functions in $L(u,\Omega)$. For example, the welfare change from x to y is larger than the change from a to b if and only if

$$|\phi(x)-\phi(y)|>|\phi(a)-\phi(b)| \quad \text{for all } \phi\in L(u,\Omega).$$

The above characterisation of an agent is a formal statement of the traditional approach. There are of course many alternative ways of describing an agent which are equivalent to the above description.

The essential difference between the modern analysis (Sen (1970), (1979)) and the analysis of the thirties is in the choice of "primitives". The modern analysis *begins* by assuming that an individual has a set L of utility functions. The traditional analysis begins by assuming the existence of u and Ω, and L is *derived* from these. The difference appears innocuous but a failure to appreciate this leads to confusions. For the time being we assume that an individual is simply an ordered pair (u, Ω).

Now we may introduce the two axioms of Lange. Axiom 1 asserts that individuals can compare utility levels. (Elements of R are denoted by u_i and this should not be confused with the utility function u).

Axiom 1. $\forall f\in\Omega,\ \forall u_1, u_2\in R,$

$$u_1\geqq u_2 \Leftrightarrow f(u_1)\geqq f(u_2).$$

In other words, only positive monotonic transformations are permitted. Clearly, the ordering on X generated by all the utility functions in $L(u,\Omega)$ will be the same if Axiom 1 is true (for all $x,\ y\in X,\ u(x)\geqq u(y)$ implies $f(u(x))\geqq f(u(y))$ for all $f\in\Omega$, by Axiom 1).

While the entire Hicksian consumer theory is based on ordinal utility, i.e. Axiom 1, we may in certain situations require a sharper conception of utility. For instance, one interpretation of the law of diminishing marginal utility requires first differences of utility to be comparable.[2] Axiom 2 states that only those transformations are permitted which retain the ordering over first differences given by u.

Axiom 2. $\forall f\in\Omega,\ \forall u_1, u_2, u_3, u_4\in R,$

$$u_1-u_2\geqq u_3-u_4 \Leftrightarrow f(u_1)-f(u_2)\geqq f(u_3)-f(u_4).$$

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An individual, (u, Ω), has cardinal utility if for all $f \in \Omega$, there exist scalars a and $b(b>0)$ such that for all $u_i \in R$, $f(u_i) = a + bu_i$. In this framework, Lange's theorem is valid. That is conditions (a) and (b), interpreted as Axioms 1 and 2, do imply that utility is cardinal. While Lange's theorem is valid, neither Lange nor Samuelson proved it in its complete generality. Lange's main proof requires u to be differentiable. But his (1933) geometric proof, while incomplete, is in this respect superior. Here he dispenses with differentiability. What he does use implicitly is the continuity of the utility function. But even this is dispensable.

Before proving Lange's theorem in the next section, it is worth observing a small result which many have overlooked. Axioms 1 and 2 are not two independent axioms at all. Whenever Axiom 2 is satisfied, Axiom 1 is automatically satisfied. This is easily established by assuming $u_3 = u_4$ in Axiom 2.

3. TWO FRAMEWORKS AND THEOREMS

In modern choice theory, particularly social choice, instead of characterising an individual as (u, Ω) and deriving $L(u, \Omega)$, it is typical to begin by directly specifying a set L of permitted utility functions. An individual is thought of as simply a set L. Any preference which is endorsed by all the utility functions in L is a preference of the individual.

In this framework an individual who can compare first-differences in utility is characterised by Axiom 2*.

Axiom 2*. Let $u \in L$. For all $\phi \in L$, for all $x, y, a, b \in X$

$$u(x) - u(y) \geqq u(a) - u(b) \Leftrightarrow \phi(x) - \phi(y) \geqq \phi(a) - \phi(b).$$

An L satisfying Axiom 2* may be labelled as an individual whose utility is *quasi-cardinal* of degree one (Basu, 1980). Cardinal utility, in this framework, is defined as a set L such that $\forall u, \phi \in L, \exists a, b(b>0)$ such that $\forall x \in X, \phi(x) = a + bu(x)$.

In this framework, being able to compare first-differences of utility (i.e. Axiom 2*) does not imply cardinality. This is easily demonstrated with the example in pages 67–68 of Basu (1980). If this is the implicit framework of Pareto's utility analysis then Pareto's assumptions of first-differences being comparable and utility being non-cardinal cannot be criticised as being inconsistent.

The critical difference between Axioms 2 and 2* which leads to these different implications is not difficult to isolate. Consider the function $\phi: X \to R$ in Axiom 2*. Corresponding to this we could derive a transformation f such that for all $x \in X$, $f(u(x)) = \phi(x)$. We shall refer to f as the transformation associated with ϕ. Now Axiom 2* may be expressed in the following equivalent manner. For all $x, y, a, b \in X$, $u(x) - u(y) \geqq u(a) - u(b) \Leftrightarrow f(u(x)) - f(u(y)) \geqq f(u(a)) - f(u(b))$. Now compare this to Axiom 2. f here is required to satisfy the same property as is required of f in Axiom 2. But with one major difference: in Axiom 2 f has to satisfy the property on the entire domain of R whereas here it has to satisfy the property on $u(X) = \{y \in R | y = u(x), x \in X\}$ which is a subset of R. This is the crucial difference. In the modern framework (given an arbitrarily chosen reference utility function $u \in L$) for any $\phi \in L$, the associated transformation f operates only on the relevant real numbers, namely $u(X)$. And the property that the ordering over first-differences is unchanged by transformations of the utility function is restricted to first differences of elements of $u(X)$. Thus the traditional framework could be thought of as the modern framework with the additional assumption of an unrestricted domain for the transformations.[3] It is this unrestricted domain assumption which makes the derivation of cardinal utility possible.

So in the modern framework the ability to compare first-differences of utility does not imply cardinality. What additional assumptions do we need to guarantee cardinality

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in this framework? One line of approach explored in Chapter 6 of Basu (1980) yields the result that for cardinality it is necessary (and sufficient) that all higher-ordered differences be comparable. Here we pursue a different route and that yields a more surprising result.

Theorem 1. *If $u(X)$ is a connected subset of R, then quasi-cardinality of degree one implies cardinality.*

Proof.[4] Given that $u(X)$ is a connected subset of R, $\forall x, y \in X$, $\exists t \in X$ such that

$$u(x)-u(t)=u(t)-u(y). \tag{1}$$

Let $\phi \in L$. Axiom 2* implies

$$\phi(x)-\phi(t)=\phi(t)-\phi(y). \tag{2}$$

In other words $\forall x, y \in X$, $\exists t \in X$ such that

$$u(x)+u(y)=2u(t) \tag{3}$$

$$\phi(x)+\phi(y)=2\phi(t). \tag{4}$$

Now suppose,

$$\textstyle\sum_{i=1}^{r} u(x_i) \geqq \sum_{i=1}^{r} u(y_i), \tag{5}$$

where $r=2^k$ (k is a positive integer) and x_i, $y_i \in X$, $\forall i$. Given (3), (5) implies $\exists t_i$, $s_i \in X (i=1, \ldots, r')$ such that

$$\textstyle\sum_{i=1}^{r'} 2u(t_i) \geqq \sum_{i=1}^{r'} 2u(s_i), \tag{6}$$

where $r'=2^{k-1}$. Similarly by (4),

$$\textstyle\sum_{i=1}^{r} \phi(x_i) \geqq \sum_{i=1}^{r} \phi(y_i) \tag{7}$$

is equivalent to

$$\textstyle\sum_{i=1}^{r'} 2\phi(t_i) \geqq \sum_{i=1}^{r'} 2\phi(s_i). \tag{8}$$

Thus (5) iff (6) and (7) iff (8).

Hence for any permutations $(x_1, \ldots, x_r)$ and $(y_1, \ldots, y_r)$ from X for which (5) holds, there exist permutations $(t_1, \ldots, t_{r'})$ and $(s_1, \ldots, s_{r'})$ from X such that [(8) iff (6)] implies [(7) iff (5)]. It follows from Basu (1980, p. 82) that all $(k-1)$-th order differences being comparable is equivalent to [(8) iff (6)] holding for all permutations $(t_1, \ldots, t_{r'})$ and $(s_1, \ldots, s_{r'})$ from X. Hence all $(k-1)$-th order differences being comparable implies [(7) iff (5)] for all permutations $(x_1, \ldots, x_r)$ and $(y_1, \ldots, y_r)$ from X, which is equivalent to all k-th order differences being comparable. Hence, given that first-order differences are comparable under quasi-cardinality of degree one, all k-th order differences are comparable, for all k. Thus by Theorem 6*5 of Basu (1980), utility is cardinal.[5] ||

From the discussion above it is clear that Axiom 2 is equivalent to Axiom 2* with $u(X)=R$. Hence Lange's result follows directly from Theorem 1:

Theorem 2. *If an individual, (u, Ω), satisfies Axiom 2 then his utility must be cardinal.*

Remark 1. In the light of Theorem 1, if X is a connected topological space and there exists a utility function in L which is continuous, then Axiom 2* implies that utility is cardinal.

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Remark 2. It is easy to check that if utility is cardinal then not only first-differences but n-th-order differences,[6] for all n, are comparable. Hence it is an immediate corollary of Theorem 1 that if the image of a utility function on X is an interval then to be able to compare first-differences implies an ability to compare all higher-order differences.

We have thus established Lange's result in a more general context, and have also established conditions under which the ability to compare first-differences of utility does imply cardinality, even within the modern framework.

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NOTES

1. We assume that the identity function is an element of Ω. Thus $u \in L(u, \Omega)$, i.e. u is a permitted utility function.

2. For an interpretation of the law of diminishing marginal utility that requires only ordinal utility and a restricted subset of first differences of utility to be comparable, see Mayston (1976).

3. After writing $f = f(u_i)$, Lange, Samuelson and others differentiate and consider the effect on f of arbitrary changes in the value of u_i. If f is defined on the domain $u(X)$ then of course these operations may not be permissible. Thus though $u(x)$, for all $x \in X$, may take a limited number of values, the transformation f is implicitly assumed by Lange and others to be equipped to transform any $u_i \in R$ to $f(u_i) \in R$, though such an u_i may not actually occur.

4. I am indebted to an anonymous referee for the present version of the proof. This version makes use of an earlier result of mine: Theorem 6*5 in Basu (1980). A longer proof which is independent of earlier results is available with the author.

5. Theorem 6*5 in Basu (1980) asserts that if k-th ordered differences are comparable, for all k, then utility is cardinal.

6. The meaning of n-th-order difference is obvious by induction given that a second-order difference is a difference of differences.

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